BOL OPERATORS, CARTIER CONTRACTION, AND LOCAL RIGIDITY AT ORDINARY PARTIAL WEIGHT ONE ON HILBERT MODULAR SURFACES

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Abstract. Let F be a real quadratic field and let $p \geq 5$ split in F. We study ordinary Hilbert modular forms near the partially singular weight $(1, k)$, where the usual BGG operator degenerates in one factor. We first identify the classical singular differential operator on the flag line as the second-order Bol operator and prove that it admits no factorization through first-order operators in the algebraic vector-bundle category, so a first-order singular operator is forced to live on a period or jet extension. We then develop the two-degeneracy derived old quotient and prove a unit-robust Cartier trace theorem for partial-Frobenius branches, allowing unit distortions, tangential variables, and matrix coefficients; combined with filtered and mixed filtration–adic erasure criteria, this gives the local and formal mechanism required to eliminate the non-old ordinary contribution. We further isolate a diagonal compactness theorem for two-term defect-one patching data, producing the patched complex, action, and augmentation from finite-level data alone, together with seven local rigidity criteria for finite flat Hecke algebras and componentwise support theorems.

The global partial-weight-one classicality and one-ring conclusions are conditional on six explicit inputs, stated in three groups — period-geometric, Goren–Oort and finite-level patching, and transversality and component support — each in the smallest form our theorems permit and each accompanied by evidence and precise obstructions. We prove the resulting implications in an acyclic order, identify the remaining constructions exactly, and close with avenues for further research. Ancillary computations check finite truncations of the key formulas but are used in no proof.

Contents

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1. Introduction

1.1. **The singular problem.** Let F be real quadratic and suppose $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$. At weight $(1,k)$, with $k \geq 2$, the $\mathfrak{p}_1$-factor is singular while the $\mathfrak{p}_2$-factor is regular. In the singular direction the positive-order BGG differential disappears; on the Galois side the two Hodge–Tate weights coincide. A satisfactory theory must therefore replace the missing BGG differential, compare the replacement with classical coherent cohomology, control the two degeneracy maps at Iwahori level, patch coherent cohomology in a two-degree range, and rule out ramification over the one-dimensional weight line.

Three distinctions govern the paper. First, integral residual-local deformation rings and characteristic-zero completed local rings at an E-valued point are different objects. A tangent computation at one point cannot prove modularity of every integral lift. Second, the old quotient at Iwahori level is generated by two degeneracy maps; its derived replacement is the cone of both maps, not the cone of a single pullback. Third, the classical and infinite-level singular operators have different orders: on the flag line the canonical singular operator is the *second-order* Bol operator, and a *first-order* singular operator can only exist on a period or jet extension, never in the classical vector-bundle category. The third distinction is a theorem of this paper (Theorem 3.2 and Proposition 3.4), and it corrects a first-order formulation that earlier versions of this project carried as a hypothesis.

1.2. **Status convention.** Every theorem, proposition, lemma, corollary, and criterion in this paper is proved from its displayed hypotheses. Statements not proved here occur only in Conjectures A–F and in the research problems of Section 12. In particular, the word "conditional" is not confined to a footnote: it appears in the abstract, in the main synthesis theorem, and in the dependency diagram. No conjecture is used in the proof of any unconditional statement, and the conditional implications form an acyclic graph.

1.3. **Unconditional core.** The following theorem is a compact index of the unconditional results; the precise statements and proofs appear in the indicated sections.

Theorem 1.1 (Unconditional core). *The following hold.*

(i) *Raw first-order partial operators have no canonical iterate (Proposition 3.1). On the flag line $\mathbf{P}^1$, for each $n \geq 0$ there is a unique* SL_2*-equivariant differential operator $\mathcal{O}(n) \to \mathcal{O}(-n-2)$ of order $n+1$ with unit principal symbol, the Bol operator; there is no nonzero such operator of order at most n; and for $n = 1$ the second-order Bol operator admits no factorization into two global first-order operators through any algebraic vector bundle (Theorem 3.2, Propositions 3.3 and 3.4).*

(ii) *The partial analytic de Rham complex on shrinking rigid discs is exact, kernels of powers of the partial derivations with Banach coefficients are the expected polynomial spaces, and the rectangular complex of the singular second-order and regular order-$(k-1)$ operators is exact with the expected bidegree-rectangle kernel (Theorem 4.2, Proposition 4.3, Theorem 4.4).*

(iii) *The two-degeneracy old cone is functorial for strict coefficient triangles and carries Hecke endomorphisms induced by chain-level intertwining homotopies. A Hecke endomorphism whose separable term is null-homotopic and whose residual term lowers a finite filtration in the homotopy category — or merely lowers it modulo p — acts topologically nilpotently on cohomology, so every ordinary projector kills the cone (Proposition 5.3, Corollary 5.5, Theorem 5.6, Theorem 5.11, Theorem 5.14).*

(iv) *The normalized trace on a partial-Frobenius branch is integral, strictly lowers transverse pole order, has the expected unit leading term on resonant graded pieces, and kills the ordinary projector; the same holds for coefficient bundles (Theorems 6.1 and 6.2).*

(v) *Two-term pigeonhole patching: from finite-level minimal complexes with adically continuous actions and complex-level augmentations, the patched complex, patched action, and augmentation isomorphism* exist*; weak data upgrade automatically; and in a defect-one situation the expected depth, freeness, and faithfulness conclusions follow from the dimension identity and nonvanishing (Theorem 7.9, Proposition 7.10, Theorem 7.2, Proposition 7.3, Corollary 7.11). For field-valued automorphy, a perfect complex with nonzero derived fibre at every minimal prime already forces $R_{\mathrm{red}} \cong \mathbb{T}_{\mathrm{red}}$ (Theorem 7.16).*

(vi) *Relative tangent vectors are fixed-weight adjoint Selmer classes, detected by finitely many unramified Frobenius trace derivatives, and seven rigidity criteria — the last admitting two equivalent fibre reformulations*

— each force a finite flat local algebra to equal its base (Theorems 8.1 and 8.6, Corollary 8.10, Proposition 8.17). Componentwise support criteria, maximal Cohen–Macaulay descent, and a dense regular-weight support theorem control the integral question (Theorems 9.1, 9.2, 9.3, Proposition 9.5).

1.4. **Conditional synthesis.** Fix an ordinary, p-distinguished E-valued eigensystem x of weight $(1, k)$ with absolutely irreducible attached representation ρ_x, in the setting of §2.1 and under Standing Hypothesis 2.1, which specifies the deformation problem and hence exactly which lifts part (c) quantifies over. The six open inputs are stated precisely in Section 10.

Theorem 1.2 (Conditional synthesis)**.**

(a) *Assuming Conjectures C, D, and E, every map in*

$$\Lambda_x \longrightarrow R_x \longrightarrow A_{\mathrm{full},x} \longrightarrow A_{\mathrm{old},x}$$

is an isomorphism. Hence the ordinary full-Iwahori eigenvariety is smooth of dimension one at x and the weight map is a local isomorphism there.

(b) *Assuming additionally Conjectures A and B, the classical weight-$(1, k)$ subspace of the localized ordinary completed cohomology at x is*

$$\ker(\mathcal{N}_{\tau_1}) \cap \ker(D_{\tau_2,k}).$$

(c) *Assuming additionally the integral component-and-control package of Conjecture F, the corresponding integral deformation-to-Hecke map is an isomorphism on every relevant component; consequently every lift satisfying the stated global and local deformation conditions is modular, with the local–global compatibility included in that conjectural package.*

The proof is formal once the six inputs are available and is given in Section 11. The theorem is deliberately local in parts (a)–(b) and integral in part (c). This prevents a tangent calculation at x from being promoted silently to an all-components modularity theorem. For the reduced, field-valued half of part (c), a strictly weaker support input suffices: Proposition 10.3 shows that finite flatness over the weight algebra together with automorphy above a Zariski-dense set of regular weights already forces support at every minimal prime, hence $R_{\mathrm{red}} \cong \mathbb{T}_{\mathrm{red}}$.

1.5. **Relation to existing work.** The operator $\mathcal{B}_n$ of Theorem 3.2 is classical: Bol's identity for automorphic factors goes back to [4], and the operator is the SL_2-equivariant germ of the theory of Eichler integrals. What we use here are the uniqueness, the order lower bound, and the factorization obstruction, which together locate the singular partial operator outside the classical category.

Pan constructs Fontaine-type differential operators on completed cohomology of modular curves and proves a regular de Rham classicality theorem

[16, 17]. Rodríguez Camargo computes the geometric Sen operator for general Shimura varieties [18]. Jiang constructs differential operators and a BGG–Fontaine complex in a general locally analytic setting, while the general automorphic properties are formulated conjecturally [14]; Jiang's Hilbert classicality theorem concerns regular parallel weights [13]. De Gaay Fortman, van Hoften, and Howe construct p-adic Maass–Shimura operators on μ-ordinary Igusa varieties [7]. These works provide compelling models for Conjectures A and B, but do not state the singular nonparallel comparison used here.

For the arithmetic side, higher Hida complexes for Hilbert modular varieties are constructed by Grossi and, in the quadratic one-prime-Iwahori setting, by Grossi–Loeffler–Zerbes [11, 10]. Calegari–Geraghty provide the patching framework for cohomology in a range [5]; our pigeonhole patching theorem is an abstract two-term instance of that mechanism, stated so that all patched-level clauses become theorems. Diamond's saving trace controls degeneracy maps and dualizing sheaves at Iwahori level [8]; Tian–Xiao and Diamond–Kassaei supply Goren–Oort and minimal-cone geometry [20, 9]. These results strongly motivate Conjectures C and D, but the exact filtered endomorphism of the two-degeneracy old cone and the singular finite-level patching datum are not presently among their stated theorems.

The eigencurve results of Bellaïche and Dimitrov, together with the ramified real-multiplication examples studied by Betina, show both why local transversality is plausible under suitable regularity and why it cannot be a purely formal consequence of finite flatness [2, 3]. The local counterexample in Section 8 records the same obstruction in commutative-algebra form. Scholze's acyclicity theorems for completed structure sheaves on affinoid perfectoids [19] are the expected bottom-stage input for the Čech acyclicity required by Conjecture A, and Hida's theory of congruence modules [12] underlies the congruence-ideal route to Conjecture E.

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2. Setup, two local categories, and support-safe notation

Let $E/\mathbb{Q}_p$ be finite with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\mathbb{F}$.

2.1. **Standing geometric and arithmetic setup.** The following objects are fixed throughout; they are specified here by reference rather than reconstructed, and every later statement that uses them quantifies over exactly these choices.

(S1) F is real quadratic with $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$, $p \geq 5$; τ_1, τ_2 are the two embeddings, matched to $\mathfrak{p}_1, \mathfrak{p}_2$ through a fixed isomorphism $\overline{\mathbb{Q}}_p \cong \mathbf{C}$. Put $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$, so that $G(\mathbb{Q}_p) \cong \mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{Q}_p)$ via the two split primes.

(S2) X denotes the Hilbert modular surface of a fixed neat tame level that is spherical at p, and Y the corresponding surface with Iwahori level at $\mathfrak{p}_1$; toroidal and minimal compactifications are fixed once and for all, and all coherent complexes below are cusp complexes on the toroidal compactification (twists by the boundary ideal), so that boundary contributions appear only through the boundary complexes treated in Proposition 4.10.

(S3) $\omega^{(a,b)}$ is the automorphic line bundle of weight (a, b), with $\omega^{(2,2)}$ matched to the dualizing data by the partial Kodaira–Spencer isomorphisms $\Omega^1_{\tau_i} \cong \omega^{(2,0)}_{\tau_i}$-type normalizations; the precise normalization of each partial Kodaira–Spencer map is the one fixed in the saving-trace framework [8], and the symbol computations of Section 3 are stated relative to it.

(S4) At infinite level, $\omega^{(a,b)}_{\mathrm{la}}$ denotes the locally analytic completed sheaf attached to $\omega^{(a,b)}$ on the ordinary locus of the perfectoid Hilbert modular surface, presented as a sectionwise filtered colimit of Banach stages over each member of a fixed finite Hodge–Tate chart cover; cohomology is continuous Čech cohomology of that cover, which Proposition 3.11 compares with the stages. This is the setting of [16, 17, 18, 14], to which we refer for the construction of the period sheaves and the Hodge–Tate period map.

(S5) $\delta_1, \delta_2 : C_X(\mathcal{F}) \to C_Y(\mathcal{F})$ are the two normalized degeneracy maps at $\mathfrak{p}_1$ on the coherent complexes of (S2), and $U_{\mathfrak{p}_1}$ is the normalized Iwahori–Hecke correspondence, acting on complexes through the chain-level data of Section 5.

(S6) The ordinary projector attached to an operator U acting on a module or on cohomology is $e_U = \lim_{n\to\infty} U^{n!}$, whenever this limit exists in the topology specified in the statement at hand (the I-adic topology of Theorem 5.11, the p-adic topology on finite $\mathcal{O}$-modules, or the norm topology of Proposition 5.15); "ordinary part" means the image of e_U.

(S7) Hecke algebras are always *faithful images*: for a fixed commutative polynomial algebra $\mathbb{H}$ of Hecke operators (unramified away from the level, together with $U_{\mathfrak{p}_1}$ and the diamond operators), the Hecke algebra of a specified module M is the image of $\mathbb{H}$ in $\mathrm{End}(M)$. In particular $A_{\mathrm{full},x}$ and $A_{\mathrm{old},x}$ are, by definition, the images of $\mathbb{H}$ on the completed localizations at x of the specified ordinary full and old cohomology modules; equality of the modules with equivalent actions therefore implies equality of the algebras.

(S8) $\overline{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbb{F})$ is the fixed residual representation, absolutely irreducible and p-distinguished at both $\mathfrak{p}_i$, with S the finite set of ramified places together with $\{v \mid p\infty\}$; the coefficient field E is large enough that x, both residual diagonal characters at each $\mathfrak{p}_i$, and all companion eigenvalues in sight are E-rational.

Standing Hypothesis 2.1. The integral deformation problem is the fixed-determinant, minimally ramified (or fixed tame type) ordinary problem for $\overline{\rho}$: lifts are required to be ordinary at $\mathfrak{p}_1$ and $\mathfrak{p}_2$ with the prescribed ordinary characters, of the fixed determinant, and of the fixed inertial types at the places of $S \setminus \{v \mid p\}$. We assume $\overline{\rho}$ is adequate in the sense required by the Taylor–Wiles–Kisin method for GL_2/F at $p \geq 5$, that the problem is representable by $R^{\mathrm{int}}_{\overline{\rho}}$ [15], and that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible. Part (c) of the conditional synthesis quantifies over exactly the lifts of this problem.

Notation index.

$\Lambda^{\mathrm{int}} = \mathcal{O}[[T]]$, $R^{\mathrm{int}}_{\overline{\rho}}$	integral weight algebra; integral deformation ring
$\Lambda_x \cong E[[t]]$, R_x	completed generic-fibre rings at x (§2.3)
$A_{\mathrm{full},x}$, $A_{\mathrm{old},x}$	completed local Hecke images at x, per (S7)
$\mathcal{B}_n$	Bol operator $\mathcal{O}(n) \to \mathcal{O}(-n-2)$ (Theorem 3.2)
$\mathcal{N}_{\tau_1}$, $D_{\tau_2,k}$	conjectural singular operator; regular BGG operator
$C_{\mathrm{old}}(\mathcal{F})$	two-degeneracy old cone (Definition 5.1)
U, e_U	normalized Hecke operator; ordinary projector, per (S6)
S_∞, S_N, $\mathfrak{a}$, J_N	diamond algebras and congruence ideals (§7)
R^{c}, D_N, D_0, M^1_∞	patching candidate; finite-level/patched complexes
$W = \mathrm{ad}^0\,\rho_x$, $\mathcal{L}$, $\mathcal{L}^{\mathrm{var}}$	adjoint module; fixed-/variable-weight Selmer structures
$\mathfrak{D}^{\mathrm{K}}_{A/\Lambda}$	Kähler different $\mathrm{Fitt}^0_A(\Omega_{A/\Lambda})$

2.2. **Integral residual-local objects.** An integral weight algebra and residual deformation ring have the form

$$\Lambda^{\mathrm{int}} = \mathcal{O}[[T]], \qquad R^{\mathrm{int}}_{\overline{\rho}},$$

and have residue field $\mathbb{F}$ under the usual representability hypotheses [15]. Integral Hecke algebras localized at the residual maximal ideal belong to this category. A theorem about every integral lift of $\overline{\rho}$ requires an integral $R = \mathbb{T}$ theorem and support on every relevant component.

2.3. **Completed generic-fibre objects at a point.** Let x be an E-valued point of the generic fibre. Put

$$\Lambda_x := (\widehat{\Lambda^{\mathrm{int}}[1/p]})_{\mathfrak{m}_x} \cong E[[t]], \qquad R_x := (\widehat{R^{\mathrm{int}}_{\overline{\rho}}[1/p]})_{\mathfrak{m}_x}.$$

The rings $A_{\mathrm{full},x}$ and $A_{\mathrm{old},x}$, when defined, are complete local E-algebras with residue field E. Tangent–Selmer comparisons at ρ_x concern these rings, not the entire integral residual-local ring.

Lemma 2.2 (Support-safe completion)**.** *Let A be noetherian, let $I \subset A$, and let $\mathfrak{p} \in \operatorname{Spec} A$. Then*

$$(A/I)_{\mathfrak{p}} = 0 \iff I \not\subseteq \mathfrak{p}.$$

If $I \subseteq \mathfrak{p}$, then

$$\widehat{(A/I)_{\mathfrak{p}}} \cong \widehat{A_{\mathfrak{p}}}/I\widehat{A_{\mathfrak{p}}}.$$

Proof. If $I \not\subseteq \mathfrak{p}$, an element of I becomes a unit after localization, so the quotient is zero. If $I \subseteq \mathfrak{p}$, then $(A/I)_{\mathfrak{p}} = A_{\mathfrak{p}}/IA_{\mathfrak{p}}$, which is nonzero because it surjects onto the residue field. Completion is exact on finite modules over a noetherian local ring, which gives the stated isomorphism. □

Warning 2.3. The symbol $A_{\mathrm{old},x}$ is nonzero only after the old quotient ideal has been shown to lie in the maximal ideal of x. A branch theorem cannot prove that x belongs to the old support while simultaneously presupposing a nonzero old completion at x. Proposition 9.7 is the device that discharges this obligation.

3. The classical singular operator: Bol operators and their obstructions

3.1. **Typed composition.** Let $(X, \mathcal{O}_X)$ be a ringed space, let L_a be line bundles, and let

$$N_a : L_a \longrightarrow L_{a-2} \otimes \Omega$$

be an additive differential operator of order at most one.

Proposition 3.1 (Typed-composition obstruction)**.** *The expression $N_{a-2} \circ N_a$ is not defined from the displayed data. An order-two composite requires an extension*

$$\widetilde{N}_{a-2} : L_{a-2} \otimes \Omega \longrightarrow L_{a-4} \otimes \Omega^{\otimes 2}$$

compatible with the symbol of N_{a-2}. Any two such extensions with the same symbol differ by an $\mathcal{O}_X$-linear map.

Proof. The target of N_a is not the domain of N_{a-2}. Moreover, a positive-order operator cannot be tensored naively with 1_Ω: the two representatives $(fs) \otimes \eta = s \otimes f\eta$ would be sent to expressions differing by the principal symbol applied to df. A connection or first-jet extension supplies the missing correction. The difference of two extensions with equal symbol has zero symbol and is therefore order zero. □

3.2. **Bol operators.** We work on $\mathbf{P}^1$ over a field K of characteristic zero, with the two standard charts U_0, U_∞, coordinates z on U_0 and $w = z^{-1}$ on U_∞, and, for each integer n, frames e_0, e_∞ of $\mathcal{O}(n)$ with $e_\infty = z^n e_0$. A section is thus a pair of regular functions $f(z)$, $g(w)$ with $g(w) = w^n f(w^{-1})$ on the overlap. Recall $T_{\mathbf{P}^1} \cong \mathcal{O}(2)$ and $\partial_w = -z^2 \partial_z$ on the overlap.

The operator in the next theorem, and the gluing identity behind it, are classical: they go back to Bol [4] and underlie the theory of Eichler integrals. We include the short proof to fix normalizations and because the application needs precisely the uniqueness and the order lower bound; the factorization obstruction of Proposition 3.4 below appears to be new.

Theorem 3.2 (Bol operator)**.** *For every integer $n \geq 0$ there is a unique global differential operator*

$$\mathcal{B}_n : \mathcal{O}_{\mathbf{P}^1}(n) \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-n-2)$$

of order at most $n+1$ whose principal symbol is the canonical trivializing section of $\mathcal{H}om\big(\mathcal{O}(n), \mathcal{O}(-n-2) \otimes \operatorname{Sym}^{n+1} T_{\mathbf{P}^1}\big)$ normalized to be 1 on the chart U_0. On the two standard charts,

$$\mathcal{B}_n(fe_0) = f^{(n+1)}e_0', \qquad \mathcal{B}_n(ge_\infty) = (-1)^{n+1}g^{(n+1)}e_\infty',$$

where e_0', e_∞' are the corresponding frames of $\mathcal{O}(-n-2)$. The operator is SL_2*-equivariant, and there is no nonzero global operator $\mathcal{O}(n) \to \mathcal{O}(-n-2)$ of order at most n.*

Proof. Symbol bundle. Since $\operatorname{Sym}^{n+1} T \cong \mathcal{O}(2n+2)$, the displayed Hom sheaf is $\mathcal{O}(-n-2+2n+2-n) = \mathcal{O}$; with respect to the frames ∂_z^{n+1}, ∂_w^{n+1} and the chosen line-bundle frames its transition function is $z^{-n-2} \cdot (-z^2)^{n+1} \cdot z^{-n} = (-1)^{n+1}$. The pair $(1, (-1)^{n+1})$ is therefore a global section, the canonical trivialization normalized on U_0.

Gluing. For every integer m, a direct computation of falling factorials gives (3.1)

$$\tfrac{d^{n+1}}{dw^{n+1}}\big(w^n(w^{-1})^m\big) = \Big[\prod_{j=0}^{n}(n-m-j)\Big]w^{-m-1} = (-1)^{n+1}\Big[\prod_{j=0}^{n}(m-j)\Big]w^{-m-1},$$

where the second equality substitutes $i = n-j$ and extracts one sign per factor. The right side is $(-1)^{n+1}w^{-n-2} \cdot \big(f^{(n+1)}\big)(w^{-1})$ for $f(z) = z^m$. Thus the identity

$$\frac{d^{n+1}}{dw^{n+1}}\big(w^n f(w^{-1})\big) = (-1)^{n+1}w^{-n-2}f^{(n+1)}(w^{-1})$$

holds for all monomials, hence for all polynomials by linearity. Both sides are differential operators in f of order at most $n+1$ with coefficients regular on the overlap, and a differential operator $\sum_{j \leq n+1} a_j \partial^j$ vanishing on $1, z, \ldots, z^{n+1}$ has $a_0 = a_1 = \cdots = a_{n+1} = 0$, by evaluating on these monomials in increasing order. Hence the identity holds on all local sections of the structure sheaf over the overlap. Since $e_\infty' = z^{-n-2}e_0'$, this is exactly the compatibility required for the two chart formulas to glue to a global operator, whose principal symbol is the section normalized above.

Vanishing in order at most n. Filter the sheaf of differential operators by order. The jth symbol quotient is

$$\mathcal{H}om\big(\mathcal{O}(n), \mathcal{O}(-n-2) \otimes \operatorname{Sym}^j T\big) \cong \mathcal{O}(2j-2n-2),$$

which has negative degree for $0 \leq j \leq n$ and hence no nonzero global sections. Ascending induction on the order along the symbol exact sequences gives $H^0\big(\mathbf{P}^1, \mathrm{Diff}^{\leq n}(\mathcal{O}(n), \mathcal{O}(-n-2))\big) = 0$.

Uniqueness and equivariance. Two order-$(n+1)$ operators with the same principal symbol differ by an operator of order at most n, which vanishes by the previous paragraph. An element of SL_2 acts on $\mathbf{P}^1$ preserving $\mathcal{O}(n)$,

$\mathcal{O}(-n-2)$, and the canonical symbol section; conjugation by it therefore produces another operator of order $n+1$ with the same symbol, which equals $\mathcal{B}_n$ by uniqueness. □

The singular partial-weight-one shape is the case $n = 1$: the canonical classical operator is the *second-order* map

$$\mathcal{B}_1 : \mathcal{O}(1) \longrightarrow \mathcal{O}(-3), \qquad f \longmapsto f''.$$

The next two propositions show that a first-order replacement is not merely unknown; it does not exist in the classical category, even through an auxiliary bundle.

Proposition 3.3 (No first-order avatar)**.** *Over a field of characteristic zero, every global differential operator of order at most one $\mathcal{O}_{\mathbf{P}^1}(1) \to \mathcal{O}_{\mathbf{P}^1}(-3)$ is zero.*

Proof. This is the case $n = 1$ of the vanishing step in Theorem 3.2: the order-0 and order-1 symbol quotients are $\mathcal{O}(-4)$ and $\mathcal{O}(-2)$, both without global sections. □

Proposition 3.4 (No algebraic first-order factorization)**.** *There is no algebraic vector bundle $\mathcal{E}$ on $\mathbf{P}^1$ and no pair of global differential operators of order at most one*

$$\mathcal{O}(1) \xrightarrow{D_1} \mathcal{E} \xrightarrow{D_2} \mathcal{O}(-3)$$

whose composite has the nonzero second-order symbol of $\mathcal{B}_1$.

Proof. By Birkhoff–Grothendieck, $\mathcal{E} \cong \bigoplus_i \mathcal{O}(a_i)$. The order-two symbol of the composite is the composite of the two principal symbols, summed over the summands of $\mathcal{E}$. The ith first symbol is a global map $\mathcal{O}(1) \to \mathcal{O}(a_i) \otimes T = \mathcal{O}(a_i + 2)$, i.e. a section of $\mathcal{O}(a_i + 1)$, which vanishes unless $a_i \geq -1$; the ith second symbol is a global map $\mathcal{O}(a_i) \to \mathcal{O}(-3) \otimes T = \mathcal{O}(-1)$, i.e. a section of $\mathcal{O}(-1 - a_i)$, which vanishes unless $a_i \leq -1$. Only summands with $a_i = -1$ can contribute.

For such a summand, the first symbol is a section of $\mathcal{O}(0)$, a nonzero constant c; after rescaling, the projection of D_1 to that summand is a first-order operator $\mathcal{O}(1) \to \mathcal{O}(-1) = \mathcal{O}(1) \otimes \Omega^1_{\mathbf{P}^1}$ with symbol the identity: an algebraic connection ∇ on $\mathcal{O}(1)$. No such connection exists, by the Atiyah–Weil obstruction [1]: a line bundle carrying an algebraic connection has vanishing Atiyah class, which for a line bundle on a curve in characteristic zero is its degree. Concretely, in the two standard trivializations the connection forms $\omega_0 = a(z)\,dz$ and $\omega_\infty = b(w)\,dw$, with a, b polynomials, would satisfy $\omega_0 - \omega_\infty = d\log z = dz/z$ on the overlap. Computing residues at $z = 0$: the form ω_0 is regular there, and $\omega_\infty = -\sum_k b_k z^{-k-2}\,dz$ has no z^{-1}-term, so both residues vanish, while $\mathrm{res}_{z=0}(dz/z) = 1$ — a contradiction. This rules out every factorization. □

Remark 3.5. Propositions 3.3 and 3.4 are global statements. On an affine chart the bundles trivialize and ordinary first-order operators exist. Thus a *first-order* singular partial operator, if it exists, must use infinite-level period geometry — a jet, period, or derived extension — rather than descend to the flag variety; and by Lemma 3.7 below, whenever the composite of two such first-order period maps descends with the correct symbol, that composite *is* the Bol operator. This is the precise sense in which Conjecture A is forced to live at infinite level, with $\mathcal{B}_1$ as its classical shadow.

Lemma 3.6 (Frame-and-symbol determination)**.** *Let L, M be line bundles on a ringed space and let $D, D' : L \to M$ be order-≤ 1 differential operators. On an open set where L has frame e, if $\sigma_D = \sigma_{D'}$ and $D(e) = D'(e)$, then $D = D'$.*

Proof. The difference $D - D'$ has zero principal symbol and is therefore $\mathcal{O}$-linear. Since it vanishes on the frame, it vanishes on every section fe. □

Lemma 3.7 (Composition-symbol rigidity)**.** *Let $L \xrightarrow{D_1} \mathcal{E} \xrightarrow{D_2} M$ be first-order operators, where $\mathcal{E}$ may be a period or jet extension. The order-two principal symbol of D_2D_1 is the composite of the two first symbols. In particular, if $L = \mathcal{O}(1)$, $M = \mathcal{O}(-3)$, the composite descends to a global operator on $\mathbf{P}^1$, and its second symbol is the canonical unit symbol, then $D_2D_1 = \mathcal{B}_1$.*

Proof. In local coordinates write $D_i = A_i\partial + B_i$ with A_i, B_i linear over the base. The coefficient of ∂^2 in the composite is A_2A_1, the composite of the symbols. For the final assertion, the difference $D_2D_1 - \mathcal{B}_1$ is a global operator of order at most one from $\mathcal{O}(1)$ to $\mathcal{O}(-3)$, hence zero by Proposition 3.3. □

3.3. **Three formal reductions for the period-side operator.**

Lemma 3.8 (Difference of multiplicative sections)**.** *Let*

$$0 \longrightarrow J \longrightarrow \mathcal{B} \xrightarrow{\theta} \mathcal{A} \longrightarrow 0$$

be an extension of sheaves of E-algebras with $J^2 = 0$, and let $\sigma_1, \sigma_2 : \mathcal{A}_0 \to \mathcal{B}$ be two E-algebra sections over a subalgebra $\mathcal{A}_0 \subseteq \mathcal{A}$. Then $D = \sigma_1 - \sigma_2$ takes values in J and is an E-derivation. It therefore factors uniquely through an $\mathcal{A}_0$-linear map $\Omega_{\mathcal{A}_0/E} \to J$; if $\mathcal{A}_0$ is generated over a D-killed subalgebra by one element z, then D is determined by the single value $D(z)$.

Proof. The square-zero ideal J is an $\mathcal{A}$-module because two lifts of an element of $\mathcal{A}$ act identically on J. Since $\theta\sigma_1 = \theta\sigma_2$, D takes values in J. For $f, g \in \mathcal{A}_0$,

$$\begin{aligned} D(fg) &= \sigma_1(f)\sigma_1(g) - \sigma_2(f)\sigma_2(g) \\ &= \sigma_2(f)D(g) + D(f)\sigma_2(g) + D(f)D(g). \end{aligned}$$

The last term is zero and the first two terms are $fD(g) + gD(f)$ under the canonical $\mathcal{A}$-module structure on J. The universal property of Kähler differentials gives the factorization; the Leibniz rule then computes D on every polynomial in z over the killed subalgebra, and by continuity on all of $\mathcal{A}_0$ in the topological setting. □

Proposition 3.9 (Čech-cone comparison)**.** *Let $\Phi : K^\bullet \to L^\bullet$ be a morphism of bounded complexes of sheaves on a finite cover $\mathfrak{U}$. Suppose every term of $K^\bullet$, $L^\bullet$, and* $\mathrm{Cone}(\Phi)$ *is Čech-acyclic on every finite intersection of $\mathfrak{U}$, and the sectionwise map on every such intersection is a quasi-isomorphism. Then $R\Gamma(\Phi)$ is a quasi-isomorphism.*

Proof. The Čech total complex of $\mathrm{Cone}(\Phi)$ computes its derived global sections. Filter it by Čech degree. On the first page, each column is the cohomology of the sectionwise cone on a finite intersection, hence zero because the cone of a quasi-isomorphism is acyclic. The filtration is finite in each total degree, so the total complex is acyclic, which is equivalent to the claim. ☐

Proposition 3.10 (Equivariance from uniqueness)**.** *Let V be an object of a ringed site, let L, M be $\mathcal{O}_V$-modules, and let a group G act compatibly on the situation so that conjugation $D \mapsto g \cdot D$ preserves order-≤ 1 operators and fixes a symbol σ. Suppose the set $\mathcal{D}_\sigma$ of global order-≤ 1 operators $L \to M$ with symbol σ is nonempty and is a torsor under $H := H^0(V, \mathcal{H}om(L, M))$. Then:*

(i) *for $D \in \mathcal{D}_\sigma$, the function $c_D(g) := g \cdot D - D$ takes values in H and satisfies $c_D(gh) = g \cdot c_D(h) + c_D(g)$;*
(ii) *the class $[c_D] \in H^1(G, H)$ is independent of D;*
(iii) *a G-equivariant member of $\mathcal{D}_\sigma$ exists if and only if $[c_D] = 0$;*
(iv) *if some $D_0 \in \mathcal{D}_\sigma$ is the unique member satisfying a G-invariant condition, then D_0 is G-equivariant.*

Proof. (i) Both $g \cdot D$ and D lie in $\mathcal{D}_\sigma$, so their difference lies in H. Conjugation is additive on differences, and

$$c_D(gh) = g \cdot (h \cdot D) - g \cdot D + g \cdot D - D = g \cdot c_D(h) + c_D(g).$$

(ii) If $D' = D + a$ with $a \in H$, then $c_{D'}(g) = c_D(g) + (g \cdot a - a)$, a coboundary shift. (iii) If $c_D(g) = g \cdot a - a$, then $D - a$ is equivariant; conversely an equivariant member has vanishing cocycle. (iv) For each g, the operator $g \cdot D_0$ satisfies the transported condition, which equals the original by invariance; uniqueness gives $g \cdot D_0 = D_0$. ☐

Proposition 3.11 (Filtered-colimit Čech acyclicity)**.** *Let $\mathfrak{U} = \{U_1, \ldots, U_s\}$ be a finite cover of V and $(F_\alpha)_{\alpha \in I}$ a filtered system of abelian sheaves with colimit F, such that for every nonempty finite intersection U_J the natural map $\varinjlim_\alpha F_\alpha(U_J) \to F(U_J)$ is an isomorphism. If every F_α has vanishing positive Čech cohomology on $\mathfrak{U}$ (respectively on every finite intersection), then so does F.*

Proof. Each degree of the Čech complex is a finite product of section groups, and finite products commute with filtered colimits of abelian groups; by hypothesis so do sections on each intersection. Hence $\check{C}^\bullet(\mathfrak{U}, F) \cong \varinjlim_\alpha \check{C}^\bullet(\mathfrak{U}, F_\alpha)$ as complexes. Filtered colimits are exact, so cohomology passes to the colimit and vanishes in positive degrees. The refined statement is the same argument on each intersection. □

Remark 3.12. Proposition 3.11 reduces the Čech acyclicity of a locally analytic sheaf presented as a sectionwise filtered colimit of Banach stages to stage-wise acyclicity; on affinoid perfectoid intersections the stage-wise input is the acyclicity of completed structure sheaves [19]. Only the sectionwise-colimit verification is geometry.

4. Local analytic kernels

Let E be a complete nonarchimedean field of characteristic zero. For $r > 0$, write $A(r) = E\langle r^{-1}z\rangle$ for the Tate algebra of series $\sum a_n z^n$ with $|a_n|r^n \to 0$, and let A^{la} be the filtered colimit along restriction to smaller radii. The same notation is used in several variables and for a Banach coefficient algebra B in place of E.

Lemma 4.1 (Bounded antiderivatives)**.** *Let $q \geq 1$ and $0 < r' < r$. For $f = \sum_{n\geq 0} a_n z^n \in B\langle r^{-1}z\rangle$, the formal q-fold antiderivative*

$$I_q(f) = \sum_{n\geq 0} \frac{a_n}{(n+1)(n+2)\cdots(n+q)}\, z^{n+q}$$

converges at radius r' and satisfies $\partial_z^q I_q = \mathrm{id}$ after restriction. On each fixed-radius algebra, $\ker(\partial_z^q)$ is the space of polynomials of degree less than q with coefficients in B. Consequently ∂_z^q is surjective on the direct limit of germs obtained by allowing an arbitrarily small loss of radius.

Proof. With $\lambda = r'/r < 1$, the norm of the nth term of $I_q(f)$ at radius r' is at most $\|f\|_r\,(r')^q \left|(n+1)\cdots(n+q)\right|_p^{-1}\lambda^n$, and $|(n+1)\cdots(n+q)|_p^{-1} \leq \big((n+1)\cdots(n+q)\big) \leq (n+q)^q$ grows subexponentially while λ^n decays exponentially. So $I_q(f)$ converges at radius r', with a norm bound linear in $\|f\|_r$, and termwise differentiation gives the identity. For the kernel, the z^n-coefficient of $\partial_z^q f$ is $(n+q)(n+q-1)\cdots(n+1)a_{n+q}$, whose scalar factor is a nonzero integer; so $a_m = 0$ for $m \geq q$. □

Theorem 4.2 (First-order partial analytic Poincaré lemma)**.** *Work in finitely many variables $z_1, \ldots, z_d$, and let J be a set of coordinate indices. The Koszul complex of the commuting derivations ∂_j, $j \in J$, on A^{la} is exact in positive degrees. Its degree-zero kernel is the subspace independent of all z_j, $j \in J$. Moreover,*

$$\ker(\partial_j^m) = \left\{ \sum_{a=0}^{m-1} z_j^a f_a : f_a \text{ independent of } z_j \right\}.$$

Proof. Lemma 4.1 gives the one-variable contraction after shrinking the radius. For several variables, induct on $\#J$: write a closed q-form as $\alpha + e_j \wedge \beta$ with α, β free of e_j; subtracting $d(I_j\beta)$, where I_j is the coefficientwise z_j-antiderivative, leaves a closed form whose coefficients are killed by ∂_j, i.e.

independent of z_j, and the inductive hypothesis applies within the z_j-degree-zero subcomplex, which the remaining Koszul differentials preserve. At most $\#J$ radius shrinks occur. Every positive-degree class at one radius therefore dies at a smaller radius, and filtered colimits of E-vector spaces are exact, so the direct-limit complex is exact in positive degrees. The degree-zero kernel and the $\ker(\partial_j^m)$ formula follow by comparing power-series coefficients: in characteristic zero every falling factorial attached to a term of degree at least m is nonzero. $\square$

Proposition 4.3 (Banach-valued polynomial kernels)**.** *Let W be a Banach E-space. Identify $A(\mathbf{r})\widehat{\otimes}_E W$ with the space of strictly convergent W-valued power series. The equation $\partial_j^m f = 0$ forces f to be a polynomial of degree $< m$ in z_j, with W-valued analytic coefficients in the other variables. Intersections of such kernels in several variables are the expected finite direct sums of bounded-degree monomials.*

Proof. Monomials form an orthogonal Schauder basis of $A(\mathbf{r})$, so $A(\mathbf{r})\widehat{\otimes}_E W \cong c_0(\mathbb{Z}^d_{\geq 0}, W)$ and every element has a unique expansion $f = \sum_\nu z^\nu w_\nu$ with jointly injective continuous coefficient maps. The coefficient of z^ν in $\partial_j^m f$ is $(\nu_j + m)\cdots(\nu_j + 1)\, w_{\nu + me_j}$, with nonzero integer scalar, so every coefficient of z_j-degree at least m vanishes. Applying the same argument in each variable gives the intersection statement. $\square$

4.1. **The rectangular complex.** For the singular direction we also need the second-order theory matching the Bol operator. Set $D_1 = \partial_{z_1}^2$ and $D_2 = \partial_{z_2}^{k-1}$ with $k \geq 2$, acting on the two-variable algebra A over a Banach algebra B, and write

$$P_{1,k-2}(B) = \Big\{ \sum_{i=0}^{1} \sum_{j=0}^{k-2} b_{ij} z_1^i z_2^j : \; b_{ij} \in B \Big\}.$$

Theorem 4.4 (Rectangular BGG exactness)**.** *In the direct-limit category of analytic germs obtained by allowing arbitrarily small losses of radius, the complex*

$$0 \longrightarrow P_{1,k-2}(B) \longrightarrow A \xrightarrow{(D_1, D_2)} A \oplus A \xrightarrow{(u,v) \mapsto D_2 u - D_1 v} A \longrightarrow 0$$

is exact. In particular the joint kernel of D_1 and D_2 is precisely the bidegree rectangle $P_{1,k-2}(B)$.

Proof. The joint-kernel assertion follows from Lemma 4.1 applied successively in z_1 and z_2: $\ker D_1$ consists of the elements $a_0(z_2) + z_1 a_1(z_2)$, and applying the z_2-kernel computation to a_0 and a_1 gives the rectangle. The complex is a complex because D_1 and D_2 commute.

Surjectivity of the last map: given $w \in A$, shrinking-radius surjectivity of D_2 (Lemma 4.1) provides u with $D_2 u = w$; take $v = 0$.

Exactness in the middle: suppose $D_2 u = D_1 v$. Choose f_0 with $D_1 f_0 = u$, again by Lemma 4.1 in the variable z_1. Then $D_1(v - D_2 f_0) = D_1 v - D_2 D_1 f_0 =$

$D_1 v - D_2 u = 0$, so

$$v - D_2 f_0 = a_0(z_2) + z_1 a_1(z_2) \in \ker D_1.$$

Choose b_0, b_1 with $\partial_{z_2}^{k-1} b_i = a_i$ and set $h = b_0(z_2) + z_1 b_1(z_2) \in \ker D_1$; then $D_2 h = v - D_2 f_0$, and $f = f_0 + h$ satisfies $D_1 f = u$ and $D_2 f = v$. $\square$

Proposition 4.5 (Horizontal coefficient extension)**.** *Let V be a finite projective B-module. If the two operators act only in the flag variables, then tensoring the complex of Theorem 4.4 with V over B preserves exactness. The same holds for any sheaf that is locally horizontally isomorphic to such a tensor product.*

Proof. A finite projective module is a direct summand of a finite free module, and both tensoring with a free module and passing to direct summands preserve exactness. A horizontal local isomorphism identifies the operators with the tensor-product operators, and exactness is a local statement. $\square$

Proposition 4.6 (Edge–Casimir factorization)**.** *Let $E = \partial_z$, $F_\lambda = -z^2\partial_z + \lambda z$, and $T = z\partial_z$. Then*

$$F_\lambda E = -T\,(T - 1 - \lambda).$$

On a disc containing $z = 0$, the kernel of $F_0 E$ is the affine functions $B \oplus Bz$, and the kernel of $F_{-1}E$ is the constants.

Proof. Direct calculation: $F_\lambda E(f) = -z^2 f'' + \lambda z f'$, while $-T(T-1-\lambda)f = -\big(zf' + z^2 f''\big) + (1+\lambda) z f' = -z^2 f'' + \lambda z f'$. For $\lambda = 0$ the operator is $-z^2\partial_z^2$; on a disc, $z^2 f'' = 0$ forces $f'' = 0$ away from 0 and hence identically, so the kernel is $B \oplus Bz$. For $\lambda = -1$ the operator is $-T^2$; comparing coefficients in $\sum a_n z^n$ gives $n^2 a_n = 0$ for $n \geq 1$. $\square$

Remark 4.7. Proposition 4.6 is the $\mathfrak{sl}_2$-structure at the edge of the singular weight: the singular second-order operator factors the shifted Casimir, and the two edge parameters produce exactly the two local kernels — the affine (form-plus-companion) kernel and the constant kernel — that appear as the z_1-degree filtration of the rectangle in Theorem 4.4. Appendix A records a finite polynomial verification of the displayed identity.

4.2. **Two kernel theories and the selection sequence.** The paper uses two singular kernels: the kernel of the conjectural *first-order* period operator, locally $\ker(\partial_{z_1})$, and the kernel of its descended *second-order* shadow $\mathcal{B}_1$, locally $\ker(\partial_{z_1}^2)$. These have different sizes — B versus $B \oplus Bz_1$ — and must never be conflated. The following lemma is the exact functorial comparison between them.

Lemma 4.8 (Kernel-layer selection sequence)**.** *Let V be a finite projective B-module, and let $D_1^{(1)} = \partial_{z_1}$, $D_1^{(2)} = \partial_{z_1}^2$, and $D_2 = \partial_{z_2}^{k-1}$ act horizontally on $A\widehat{\otimes}_B V$. Write $K^{(i)} = \ker(D_1^{(i)}) \cap \ker(D_2)$. Then ∂_{z_1} induces a short exact sequence*

$$0 \longrightarrow K^{(1)} \longrightarrow K^{(2)} \xrightarrow{\partial_{z_1}} K^{(1)} \longrightarrow 0,$$

natural in V and compatible with every B-linear horizontal operator commuting with ∂_{z_1} and ∂_{z_2}. Thus the second-order joint kernel is canonically an extension of the first-order joint kernel by itself: the sub is the fixed-weight layer, and the quotient is the companion layer.

Proof. By Theorem 4.4 and Proposition 4.5, $K^{(2)} = \{a_0(z_2) + z_1 a_1(z_2)\}$ with $\partial_{z_2}^{k-1} a_i = 0$ and coefficients in V, while $K^{(1)} = \{a_0(z_2)\}$. The map ∂_{z_1} sends $a_0 + z_1 a_1 \mapsto a_1$, visibly surjective onto $K^{(1)}$ with kernel $K^{(1)}$; it commutes with D_2 and with horizontal operators, and every identification is natural in V. □

Remark 4.9 (Which kernel the main theorems use). The conditional classicality statement (Theorem 1.2(b)) is a statement about the *first-order* joint kernel $\ker(\mathcal{N}_{\tau_1}) \cap \ker(D_{\tau_2,k})$, whose local model has dimension $k-1$ over the coefficient algebra. The proved comparison scaffolding of this section (Theorem 4.4, Theorem 4.11) concerns the *second-order* joint kernel, of local dimension $2(k-1)$, which Lemma 4.8 exhibits as a self-extension of the first-order kernel; the extra layer is the expected companion layer, matching the two colliding Hodge–Tate weights at $\mathfrak{p}_1$. Passing from the second-order comparison to the first-order one is *not* formal: it requires identifying the classical subspace inside the rectangle and controlling the companion layer, and this identification is part of the remaining content of Conjecture B — with the no-companion input of Conjecture E as its natural arithmetic partner. The selection sequence above is the device that makes this bookkeeping exact rather than heuristic.

Proposition 4.10 (Localized boundary acyclicity). *Let a commutative Hecke algebra $\mathbb{H}$ act on a bounded complex whose cohomology modules are finite over $\mathbb{H}$ (a boundary complex). Suppose every Hecke eigensystem occurring in any subquotient of that cohomology has residual pseudorepresentation equal to a sum of two characters $\bar{\chi}_1 + \bar{\chi}_2$. Then localization at a maximal ideal $\mathfrak{m} \subset \mathbb{H}$ whose residual two-dimensional pseudorepresentation is absolutely irreducible makes the cohomology of the boundary complex vanish.*

Proof. If some localized cohomology module were nonzero, it would be a nonzero finite module over $\mathbb{H}_{\mathfrak{m}}$, and any maximal element of its support would produce a Hecke eigensystem occurring in the boundary cohomology whose residual pseudorepresentation is that of $\mathfrak{m}$. By hypothesis that pseudorepresentation is $\bar{\chi}_1 + \bar{\chi}_2$, hence reducible, contradicting the absolute irreducibility defining $\mathfrak{m}$. □

Theorem 4.11 (Globalized rectangular comparison criterion). *Fix a finite cover $\mathfrak{U}$ of the relevant Hilbert modular completed-cohomology space and a Hecke-equivariant morphism of bounded complexes of sheaves*

$$\Phi : K^{\bullet}_{\mathrm{cl}} \longrightarrow K^{\bullet}_{\mathrm{an}},$$

from a classical coherent weight-$(1,k)$ complex to the joint-kernel complex of the descended singular operator and the regular operator, compatible with the transition maps of $\mathfrak{U}$. Assume:

(1) *on each chart the singular operator descends to a global order-two operator with the normalized partial Kodaira–Spencer symbol — hence equals $\mathcal{B}_1$ by Theorem 3.2 and Lemma 3.7 — and the regular operator is the usual order-$(k-1)$ BGG operator;*
(2) *after a finite horizontal cover, the coefficient sheaf has the finite projective horizontal form of Proposition 4.5;*
(3) *on every nonempty finite intersection of $\mathfrak{U}$, the map Φ restricts to the canonical identification of the classical coefficient module with the rectangle of Theorem 4.4, and is in particular a sectionwise quasi-isomorphism there;*
(4) *every term of $K^\bullet_{\mathrm{cl}}$, $K^\bullet_{\mathrm{an}}$, and $\mathrm{Cone}(\Phi)$ is Čech-acyclic on every nonempty finite intersection of $\mathfrak{U}$;*
(5) *the boundary complex satisfies the hypotheses of Proposition 4.10.*

Then, after non-Eisenstein localization, $R\Gamma(\Phi)$ is a quasi-isomorphism: the localized second-order joint-kernel complex is computed by the classical coherent complex.

Proof. Hypotheses (1) and (2) identify, on each chart and each finite intersection, the joint kernel with the algebraic coefficient module (Theorem 4.4, Proposition 4.5); hypothesis (3) says Φ realizes exactly this identification, so Φ is intersectionwise a quasi-isomorphism. With hypothesis (4), Proposition 3.9 makes $R\Gamma(\Phi)$ a quasi-isomorphism over the open part. Hypothesis (5) and Proposition 4.10 kill the localized boundary cone, so the statement persists after compactification and non-Eisenstein localization. □

Remark 4.12. Hypotheses (1), (2), and (5) are the three *geometric* checks — symbol identification, horizontal local triviality, and boundary control. Hypotheses (3) and (4) are the sheaf-theoretic scaffolding that any concrete construction of Φ must carry: the morphism itself, its intersectionwise behavior, and Čech acyclicity of the terms. Listing them separately makes the theorem formally complete and displays exactly what Conjecture B still has to construct. Note also that the conclusion concerns the *second-order* joint kernel; the passage to the first-order statement is governed by Lemma 4.8 and Remark 4.9.

5. The two-degeneracy derived old quotient

Let X be spherical level and Y Iwahori level at $\mathfrak{p}_1$. Abstractly, suppose a coefficient object $\mathcal{F}$ gives complexes $C_X(\mathcal{F})$ and $C_Y(\mathcal{F})$ and two compatible degeneracy maps

$$\delta_1, \delta_2 : C_X(\mathcal{F}) \longrightarrow C_Y(\mathcal{F}).$$

The geometric normalizations and line-bundle identifications required to define these maps belong to any application.

Definition 5.1 (Derived old cone)**.** The derived full-versus-old quotient is

$$C_{\mathrm{old}}(\mathcal{F}) := \mathrm{Cone}\Big(C_X(\mathcal{F})^{\oplus 2} \xrightarrow{(\delta_1,\delta_2)} C_Y(\mathcal{F}) \Big).$$

Remark 5.2. Using only one pullback map measures full level modulo one spherical image; it is not, without an additional argument, the old quotient generated by the two degeneracy maps.

Proposition 5.3 (Exact triangles for the old cone)**.** *Suppose that degreewise short exact sequences of complexes associated with $0 \to \mathcal{L} \to \mathcal{L}^+ \to \mathcal{C} \to 0$ are given at X and Y, and that the two degeneracy maps commute strictly with them at the chain level. Then*

$$C_{\mathrm{old}}(\mathcal{L}) \longrightarrow C_{\mathrm{old}}(\mathcal{L}^+) \longrightarrow C_{\mathrm{old}}(\mathcal{C}) \longrightarrow$$

is an exact triangle. If $H^i(C_{\mathrm{old}}(\mathcal{L}^+)) = 0$, then

$$H^{i-1}(C_{\mathrm{old}}(\mathcal{C})) \twoheadrightarrow H^i(C_{\mathrm{old}}(\mathcal{L})).$$

If the map

$$H^{j+1}(C_X(\mathcal{F}))^{\oplus 2} \longrightarrow H^{j+1}(C_Y(\mathcal{F}))$$

is injective, then

$$H^j(C_{\mathrm{old}}(\mathcal{F})) \cong \mathrm{coker}\big(H^j(C_X(\mathcal{F}))^{\oplus 2} \longrightarrow H^j(C_Y(\mathcal{F}))\big).$$

Proof. In degree n, the cone is $C_Y^n \oplus C_X^{n+1} \oplus C_X^{n+1}$. The strict chain-level compatibility makes the sequence of the three cones degreewise exact, hence it yields the first triangle; at the level of bare triangulated categories cones are not functorial, which is why the chain-level reading is the usable one. The associated long exact sequence gives the surjection. The defining cone triangle gives an exact sequence

$$0 \to \mathrm{coker}(u_j) \to H^j(C_{\mathrm{old}}) \to \ker(u_{j+1}) \to 0,$$

where $u_q : H^q(C_X)^{\oplus 2} \to H^q(C_Y)$; injectivity of u_{j+1} gives the last assertion. □

Lemma 5.4 (Induced endomorphism of a cone)**.** *Let $f : A^\bullet \to B^\bullet$ be a chain map, let U_A, U_B be chain endomorphisms, and let $h : A^{n+1} \to B^n$ satisfy*

$$U_B f - f U_A = dh + hd.$$

Then

$$\widehat{U}(b,a) = (U_B b + h(a), U_A a)$$

defines a chain endomorphism of $\mathrm{Cone}(f)^n = B^n \oplus A^{n+1}$*. Its homotopy class depends only on the class of h in the Hom complex.*

Proof. Using $d_{\mathrm{Cone}}(b,a) = (db + f(a), -da)$, a direct calculation shows that $d_{\mathrm{Cone}}\widehat{U} - \widehat{U} d_{\mathrm{Cone}}$ has first component $(U_B f - f U_A - dh - hd)(a)$ and second component zero. Hence it vanishes. If $h - h' = d\sigma - \sigma d$, the endomorphisms defined by h and h' differ by the cone homotopy $(b,a) \mapsto (\sigma(a), 0)$. □

Corollary 5.5 (Hecke action on the two-degeneracy cone). *Suppose U_Y is a chain endomorphism of C_Y, V_{ij} are chain endomorphisms of C_X, and there are homotopies h_i with*

$$U_Y\delta_i - \sum_j \delta_j V_{ji} = dh_i + h_i d \qquad (i = 1, 2).$$

Then C_{old} carries an induced Hecke endomorphism, well defined up to the homotopy ambiguity of Lemma 5.4.

Proof. Apply Lemma 5.4 to $f = (\delta_1, \delta_2)$, to the matrix endomorphism (V_{ij}) of $C_X^{\oplus 2}$, and to $h = (h_1, h_2)$. □

Theorem 5.6 (Filtered nilpotence in the homotopy category). *Let $C^\bullet$ have a finite filtration by subcomplexes*

$$0 = F^{a+1}C \subseteq F^aC \subseteq \cdots \subseteq F^0C = C.$$

Let U be an endomorphism of C in the homotopy category. Suppose there is an endomorphism N with $[U] = [N]$ and, for every j, a morphism $n_j : F^jC \to F^{j+1}C$ in the homotopy category such that

$$[N][\iota_j] = [\iota_{j+1}][n_j],$$

where $\iota_j : F^jC \to C$. Then $[U]^{a+1} = 0$. In particular, the action of U on every cohomology group is nilpotent and every ordinary projector obtained as a limit of positive powers of U acts as zero.

Proof. Iterating the displayed factorization gives $[N]^r[\iota_j]$ factoring through $F^{j+r}C$. Taking $j = 0$ and $r = a + 1$ gives $[N]^{a+1} = 0$ because $F^{a+1}C = 0$. Since $[U] = [N]$, the same holds for U. A null-homotopic power acts as zero on cohomology, and all sufficiently high powers are zero there. □

Lemma 5.7 (Null-homotopy as a derived obstruction). *Let A, B be cochain complexes of modules over a ring. Chain maps modulo chain homotopy form $H^0(\mathrm{Hom}^\bullet(A, B))$, and a chain map is null-homotopic exactly when its class there vanishes. If A is a bounded-above complex of projectives, the natural map $\mathrm{Hom}_K(A, B) \to \mathrm{Hom}_D(A, B)$ from the homotopy to the derived category is bijective; for perfect coherent models the obstruction is thus the vanishing of a single hypercohomology class of $R\mathcal{H}om$.*

Proof. With the differential $\partial(f) = d_Bf - (-1)^{|f|} f d_A$, degree-zero cocycles are the chain maps and degree-zero coboundaries are the null-homotopic maps. A bounded-above complex of projectives is K-projective, so $\mathrm{Hom}_K(A, -)$ inverts quasi-isomorphisms and the localization map is bijective; for perfect complexes on a scheme, $\mathrm{Hom}_D(A, B) = H^0(X, R\mathcal{H}om(A, B))$. □

Proposition 5.8 (Concentrated homotopy criterion). *Let C, D be bounded complexes of projective modules whose cohomology is concentrated in one common degree d. Then*

$$\mathrm{Hom}_K(C, D) \xrightarrow{\sim} \mathrm{Hom}(H^d(C), H^d(D)),$$

where K is the homotopy category. In particular, a chain map is null-homotopic if and only if its map on H^d is zero, and every module map on H^d is realized by a chain map, unique up to homotopy.

Proof. Bounded complexes of projectives are K-projective, so maps in the homotopy and derived categories agree (Lemma 5.7). The canonical truncations give quasi-isomorphisms $\tau_{\le d}C \to C$ and $\tau_{\le d}C \to H^d(C)[-d]$, so $C \simeq H^d(C)[-d]$ and $D \simeq H^d(D)[-d]$ in the derived category, and

$$\mathrm{Hom}_D(C, D) = \mathrm{Ext}^0(H^d(C), H^d(D)) = \mathrm{Hom}(H^d(C), H^d(D)).$$

Functoriality of the truncations identifies the bijection with $\Phi \mapsto H^d(\Phi)$. □

Proposition 5.9 (Filtered comparison)**.** *Let $\Phi : K^\bullet \to L^\bullet$ be a morphism of complexes in an abelian category, and suppose both complexes have finite decreasing filtrations by subcomplexes, preserved by Φ. If every graded map $\mathrm{gr}^j \Phi$ is a quasi-isomorphism, then Φ is a quasi-isomorphism.*

Proof. Descending induction on j, starting from the zero complexes at the top of the filtration. The commutative diagram of short exact sequences of complexes

$$0 \to F^{j+1}K \to F^jK \to \mathrm{gr}^j K \to 0, \qquad 0 \to F^{j+1}L \to F^jL \to \mathrm{gr}^j L \to 0,$$

connected by $F^{j+1}\Phi$, $F^j\Phi$, and $\mathrm{gr}^j \Phi$, induces a morphism of long exact cohomology sequences. In each degree the five lemma applies, with the four outer maps isomorphisms by the inductive hypothesis and the graded hypothesis; hence $F^j\Phi$ is a quasi-isomorphism. At $j = 0$ this is the claim. □

Proposition 5.10 (A graded criterion for homotopy lowering)**.** *Let $C^\bullet$ carry a finite filtration by subcomplexes as in Theorem 5.6, and assume each inclusion $F^{j+1}C^n \subseteq F^jC^n$ is a split monomorphism of modules in every degree n. Let N be a chain endomorphism preserving each F^jC. If the induced endomorphism of every $\mathrm{gr}^j C$ is null-homotopic, then the factorizations required in Theorem 5.6 exist, and $[N]^{a+1} = 0$.*

Proof. Fix j, write $\iota' : F^{j+1}C \to F^jC$ and $\pi_j : F^jC \to \mathrm{gr}^j C$. Because the short exact sequence $0 \to F^{j+1}C \to F^jC \to \mathrm{gr}^j C \to 0$ is degreewise split, applying $\mathrm{Hom}^\bullet(F^jC, -)$ termwise yields a short exact sequence of Hom complexes, whose degree-zero cohomology sequence reads

$$H^0\,\mathrm{Hom}(F^jC, F^{j+1}C) \xrightarrow{\iota'_*} H^0\,\mathrm{Hom}(F^jC, F^jC) \xrightarrow{(\pi_j)_*} H^0\,\mathrm{Hom}(F^jC, \mathrm{gr}^j C).$$

The class of $N|_{F^jC}$ maps under $(\pi_j)_*$ to the class of $\mathrm{gr}^j N \circ \pi_j$; note that it is this *composite with the projection*, not a degreewise component of N, that is a chain map. If s_j is a null-homotopy of $\mathrm{gr}^j N$, then $s_j\pi_j$ null-homotopes the composite, so $(\pi_j)_*[N|_{F^jC}] = 0$ and exactness provides a chain map $n_j : F^jC \to F^{j+1}C$ with $N|_{F^jC} \simeq \iota' n_j$. Post-composing with ι_j gives $[N][\iota_j] = [\iota_{j+1}][n_j]$, and Theorem 5.6 concludes. □

5.1. **Mixed filtration–adic contraction.** The following theorems remove the need for the residual term to lower the filtration on the nose: lowering modulo an ideal of definition suffices. The first is the module form; the second is the precise chain-level form, in which the homotopy data are required only on the associated graded pieces modulo I, where the Cartier engine actually produces them.

Theorem 5.11 (Mixed filtration–adic contraction, module form)**.** *Let R be a ring with an ideal I, let C be an R-module, and let*

$$C = F^0C \supseteq F^1C \supseteq \cdots \supseteq F^{a+1}C = 0$$

be a finite filtration by submodules. If an R-linear endomorphism U satisfies

$$U(F^jC) \subseteq F^{j+1}C + IC \qquad (0 \le j \le a),$$

then

$$U^{a+1}C \subseteq IC, \qquad U^{n(a+1)}C \subseteq I^nC \quad (n \ge 1).$$

If moreover C is I-adically separated, then U is topologically nilpotent and every I-adic limit of positive powers of U is zero.

Proof. By induction on j, $U^j(C) \subseteq F^jC + IC$: the base case is trivial, and $U(F^jC + IC) \subseteq F^{j+1}C + IC + I\,U(C) \subseteq F^{j+1}C + IC$ using R-linearity. At $j = a+1$ this gives $U^{a+1}C \subseteq IC$. Then, again by R-linearity, $U^{(n+1)(a+1)}C = U^{a+1}\big(U^{n(a+1)}C\big) \subseteq U^{a+1}(I^nC) = I^n\,U^{a+1}C \subseteq I^{n+1}C$. □

Theorem 5.12 (Mixed contraction, homotopy form)**.** *Let R be a ring with an ideal I, and let $C^\bullet$ be a complex of projective R-modules carrying a finite filtration by subcomplexes $0 = F^{a+1}C \subseteq \cdots \subseteq F^0C = C$ whose inclusions are split monomorphisms in every degree. Let U be a chain endomorphism preserving each F^jC termwise, and suppose that the induced endomorphism of each $\mathrm{gr}^j\,C \otimes_R R/I$ is null-homotopic. Then for every $n \ge 1$ the power $U^{(a+1)n}$ is chain homotopic to a chain map with image in $I^nC^\bullet$.*

If in addition R is noetherian, I lies in its Jacobson radical, and $C^\bullet$ is a bounded complex of finite projective modules, then U acts topologically nilpotently on every $H^i(C^\bullet)$ for the I-adic topology, and every I-adic limit of positive powers of U acts as zero.

Proof. One step. The filtration of $C/IC := C \otimes_R R/I$ is again degreewise split with graded pieces $\mathrm{gr}^j\,C \otimes R/I$, and the reduction $\overline{U}$ preserves it termwise, with null-homotopic graded pieces by hypothesis. Proposition 5.10, applied over R/I, gives $[\overline{U}]^{a+1} = 0$ in the homotopy category: there is a degree -1 map $\bar h$ on C/IC with $\overline{U}^{a+1} = d\bar h + \bar h d$. Since each C^n is projective, the composite $C^n \to C^n/IC^n \xrightarrow{\bar h} C^{n-1}/IC^{n-1}$ lifts through the surjection $C^{n-1} \twoheadrightarrow C^{n-1}/IC^{n-1}$ to a map $h : C^n \to C^{n-1}$. Set

$$V := U^{a+1} - dh - hd.$$

Then V is a chain map (a chain map minus a null-homotopic one) and $V \equiv 0$ modulo I, i.e. $V(C^\bullet) \subseteq IC^\bullet$.

Iteration. We show by induction that $U^{(a+1)n} \simeq W_n$ for a chain map W_n with $W_n(C^\bullet) \subseteq I^n C^\bullet$; the case $n = 1$ is $W_1 = V$. For the step, write $U^{(a+1)(n+1)} = U^{(a+1)n}\big(dh + hd + V\big)$. The first part is null-homotopic, because $U^{(a+1)n}$ is a chain map: $U^{(a+1)n}(dh + hd) = d\big(U^{(a+1)n}h\big) + \big(U^{(a+1)n}h\big)d$. For the second part, the inductive hypothesis gives $U^{(a+1)n} = W_n + d\sigma + \sigma d$, and since V is a chain map, $(d\sigma + \sigma d)V = d(\sigma V) + (\sigma V)d$ is null-homotopic; hence $U^{(a+1)(n+1)} \simeq W_n V =: W_{n+1}$, and by R-linearity $W_{n+1}(C) \subseteq W_n(IC) = I\,W_n(C) \subseteq I^{n+1}C$.

Cohomology. Homotopic maps agree on cohomology, so $U^{(a+1)n}$ acts on $H^i(C)$ as W_n, whose image lies in the image of $Z^i \cap I^n C^i$ in $H^i(C)$, where Z^i denotes the cycles. With R noetherian and C^i finite, the Artin–Rees lemma provides c with $Z^i \cap I^n C^i \subseteq I^{n-c} Z^i$ for $n \geq c$, so the image of $U^{(a+1)n}$ on $H^i(C)$ lies in $I^{n-c} H^i(C)$. Since $H^i(C)$ is finite and I lies in the Jacobson radical, Krull intersection makes $H^i(C)$ I-adically separated, and the powers of U tend to zero; any I-adic limit of positive powers is therefore zero. □

5.2. **The conductor triangle.** The normalization-conductor term of Conjecture C is controlled by the following standard construction, recorded here so that the conjecture can refer to a displayed object.

Proposition 5.13 (Conductor triangle)**.** *Let Z be a reduced noetherian scheme with finite normalization $\nu : \widetilde{Z} \to Z$, and set $\mathcal{Q}_Z := \operatorname{coker}(\mathcal{O}_Z \to \nu_*\mathcal{O}_{\widetilde{Z}})$. Then*

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \nu_*\mathcal{O}_{\widetilde{Z}} \longrightarrow \mathcal{Q}_Z \longrightarrow 0$$

is exact, and $\mathcal{Q}_Z$ is supported on the non-normal locus. If $\widetilde{Z} = Z_{\mathrm{sep}} \amalg Z_F$ decomposes into clopen pieces, then $\nu_*\mathcal{O}_{\widetilde{Z}} = \nu_{\mathrm{sep},*}\mathcal{O}_{Z_{\mathrm{sep}}} \oplus \nu_{F,*}\mathcal{O}_{Z_F}$. Any functor of the kernel object that is exact in the triangulated sense sends the associated distinguished triangle to a distinguished triangle whose third term is the transform of $\mathcal{Q}_Z$.

Proof. Affine-locally, a reduced noetherian ring embeds in its integral closure inside its total ring of fractions, which is what normalization is; hence $\mathcal{O}_Z \to \nu_*\mathcal{O}_{\widetilde{Z}}$ is injective and the sequence is exact by the definition of $\mathcal{Q}_Z$. Over the normal locus the normalization is an isomorphism and localization commutes with the finite pushforward, so $\mathcal{Q}_Z$ vanishes there. The decomposition statement is the additivity of ν_* on clopen pieces. A short exact sequence of coherent sheaves is a distinguished triangle in the derived category, and a triangulated functor of the kernel variable preserves distinguished triangles. □

Theorem 5.14 (Derived old-cone erasure)**.** *Let $i : O^\bullet \to C^\bullet$ and $r : C^\bullet \to O^\bullet$ be chain maps. Assume compatible Hecke endomorphisms decompose as*

$$U_C = ir + N_C, \qquad U_O = ri + N_O,$$

and that the endomorphism N_Q induced by (N_C, N_O) on $Q^\bullet = \operatorname{Cone}(i)$ satisfies one of the following:

(a) *the hypotheses of Theorem 5.12: a finite termwise-split filtration of $Q^\bullet$ preserved termwise by N_Q, graded pieces null-homotopic modulo I, bounded finite projective terms, R noetherian with I in its Jacobson radical; or*

(b) *on each cohomology module $H^n(Q^\bullet)$, the induced action of N_Q preserves a finite filtration with the containment of Theorem 5.11, and $H^n(Q^\bullet)$ is I-adically separated.*

Then the ordinary part of $H^*(Q^\bullet)$ vanishes.

Proof. With the convention $Q^n = C^n \oplus O^{n+1}$ and differential $d_Q(c, o) = (d_C c + i(o), -d_O o)$, define $h(c, o) = (0, r(c))$. Then

$$d_Q h(c, o) = (ir(c),\, -d_O r(c)), \qquad hd_Q(c, o) = (0,\, r(d_C c) + ri(o)),$$

and summing, using that r is a chain map, $(d_Q h + hd_Q)(c, o) = (ir(c), ri(o))$. Thus the factorable contribution (ir, ri) is null-homotopic on the cone, and U_Q is homotopic to N_Q; in particular the two agree on cohomology. Under (a), Theorem 5.12 makes the powers of N_Q, and hence of U_Q, tend to zero I-adically on each $H^n(Q^\bullet)$; under (b), Theorem 5.11 applies directly to the induced action. In both cases every ordinary projector — an I-adic limit of positive powers of U_Q on cohomology, as in (S6) — acts as zero. □

Proposition 5.15 (Positive-slope topological nilpotence)**.** *Let E be a field complete with respect to a nontrivial nonarchimedean absolute value with valuation v, let V be a finite-dimensional E-vector space, and let $U \in \mathrm{End}_E(V)$ be such that every eigenvalue of U in a fixed algebraic closure has positive valuation — equivalently, every nonleading coefficient of the characteristic polynomial of U lies in the maximal ideal $\mathfrak{m}_E$. Then $U^n \to 0$; consequently every limit of positive powers of U is zero. In particular, if Q is a complex whose cohomology groups are finite-dimensional E-spaces carrying U and on every $H^n(Q)$ all U-eigenvalues have positive valuation, then every ordinary projector annihilates $H^*(Q)$.*

Proof. Equivalence. Write $\chi_U(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0$. If every root has positive valuation, each a_i is $\pm$ an elementary symmetric function of the roots, of valuation at least $(d-i)\min_\lambda v(\lambda) > 0$. Conversely, if all $a_i \in \mathfrak{m}_E$ and λ is a root with $v(\lambda) \le 0$, then for $i \le d-1$, $v(a_i\lambda^i) > i\,v(\lambda) \ge d\,v(\lambda) = v(\lambda^d)$, so the ultrametric inequality contradicts $\lambda^d = -\sum_{i<d} a_i\lambda^i$.

Nilpotence. By Cayley–Hamilton, $U^d = -\sum_{i<d} a_i U^i$ with all $a_i \in \mathfrak{m}_E$. Let $M = \mathcal{O}_E \cdot 1 + \mathcal{O}_E U + \cdots + \mathcal{O}_E U^{d-1}$, a bounded finitely generated $\mathcal{O}_E$-submodule of $\mathrm{End}_E(V)$. Then $UM \subseteq M$ and $U^d M \subseteq \mathfrak{m}_E M$, so by induction $U^{dn}M \subseteq \mathfrak{m}_E^n M \to 0$; since $1 \in M$ and each U^r $(r < d)$ is bounded, $U^n \to 0$. □

Remark 5.16. Proposition 5.15 is the device that reduces ordinary acyclicity of a *regular-weight* old cone to the classical slope statement that every $U_{\mathfrak{p}_1}$-eigenvalue on the regular-weight new part has positive valuation; see clause (C5) of Conjecture C.

6. The unit-robust Cartier engine

Let $\mathcal{O}$ be a complete discrete valuation ring of mixed characteristic $(0,p)$ with $p \geq 3$, let $A = \mathcal{O}[[y_1, \ldots, y_r]]$, and put

$$R = A[[x]], \qquad S = A[[t]].$$

Let $a^{\#} : R \to S$ be the A-algebra map

$$a^{\#}(x) = u_0 t^p, \qquad u_0 \in S^{\times},$$

and assume

$$\overline{u}_0 \in (A/pA)[[t^p]]. \tag{6.1}$$

Let $b^{\#} : R \to S$ be a second A-algebra map with $b^{\#}(x) = u_1 t$, $u_1 \in S^{\times}$.

Theorem 6.1 (Unit-robust Cartier trace)**.** *The following statements hold.*

(a) *S is finite free over R, via $a^{\#}$, with basis $1, t, \ldots, t^{p-1}$, and $S[1/t] = S \otimes_R R[1/x]$.*

(b) *$\mathrm{Tr}_{S/R}(S) \subseteq pR$ and $\mathrm{Tr}_{S/R}(tS) \subseteq pxR$.*

(c) *For every $w \in S$, the operator*

$$U(g) = \frac{1}{p} \mathrm{Tr}_{S/R}\big(wt^{-1}b^{\#}(g)\big)$$

maps $R[1/x]$ to itself and maps R to R.

(d) *If $m \geq 1$ and $\kappa = \lceil (m+1)/p \rceil$, then*

$$U(x^{-m}R) \subseteq \begin{cases} x^{-(m+1)/p}R, & p \mid m+1, \\ x^{-(\kappa-1)}R, & p \nmid m+1. \end{cases}$$

In both cases the pole order drops strictly.

(e) *If $p \mid m+1$, then modulo the next lower pole step,*

$$U(x^{-m}) \equiv \big(wu_1^{-m}u_0^{(m+1)/p}\big)(0)x^{-(m+1)/p}.$$

When $w \in S^{\times}$, the coefficient is a unit of A. If $p \nmid m+1$, the induced map on the m-th pole-graded piece is zero.

(f) *On every U-stable subquotient of $x^{-a}R/R$, and more generally on every finite filtered module whose graded action satisfies the pole bounds in* (d)*, the operator U is nilpotent. Hence every ordinary projector obtained as a limit of positive powers of U acts as zero.*

Proof. Throughout, S is viewed as an R-module via $a^{\#}$, and A is a domain.

(a). We first prove generation. Since $a^{\#}(x)S = u_0 t^p S = t^p S$, write recursively

$$\sigma_k = \sum_{i=0}^{p-1} a_{k,i} t^i + t^p \tau_k, \qquad \sigma_{k+1} = u_0^{-1} \tau_k,$$

starting from $\sigma_0 = s$. Unfolding gives, for every K,

$$s = \sum_{i=0}^{p-1} \left(\sum_{k=0}^{K} a_{k,i} a^{\#}(x)^k \right) t^i + a^{\#}(x)^{K+1} \sigma_{K+1}.$$

The last term tends to zero t-adically and the inner sums converge to $a^{\#}(r_i)$ for $r_i \in R$, since $a^{\#}$ is continuous from the x-adic to the t-adic topology. Thus $s = \sum_{i<p} a^{\#}(r_i) t^i$.

For independence, suppose $\sum_{i=0}^{p-1} a^{\#}(r_i) t^i = 0$, with $r_i = \sum_{n \geq 0} c_{i,n} x^n$ not all zero. The function $(i, n) \mapsto pn + i$ is injective on $\{0, \ldots, p-1\} \times \mathbb{Z}_{\geq 0}$; choose the unique pair (i_0, n_0) with $c_{i_0,n_0} \neq 0$ minimizing $pn + i$. The term $c_{i,n} u_0^n t^{pn+i}$ contributes to the coefficient of $t^{pn_0+i_0}$ only when $pn+i \leq pn_0+i_0$, hence only for $(i, n) = (i_0, n_0)$, and that contribution is

$$c_{i_0,n_0} u_0(0)^{n_0} \neq 0,$$

because A is a domain and $u_0(0)$ is a unit. This is a contradiction; note that a general unit u_0 spreads $u_0^n t^{pn+i}$ over all degrees $\geq pn + i$, which is why the argument tracks the unique minimal pair rather than degrees modulo p. Hence $1, t, \ldots, t^{p-1}$ is an R-basis. Finally, $a^{\#}(x) = u_0 t^p$ shows that inverting x is equivalent to inverting t, with $t^{-1} = a^{\#}(x)^{-1} u_0 t^{p-1}$.

(b). Trace commutes with base change. Modulo p, put $s = t^p$ and write $\overline{u}_0 = v(s)$ using (6.1). The substitution $s \mapsto v(s)s$ has a compositional inverse because its linear coefficient is a unit. Thus the image of R/pR in S/pS is $(A/pA)[[t^p]]$. Relative to the basis $1, t, \ldots, t^{p-1}$, multiplication by $\sum_{j=0}^{p-1} f_j(t^p) t^j$ has diagonal entries only from the $j = 0$ term, all equal to $f_0(t^p)$. Its trace is therefore $pf_0(t^p) = 0$ in characteristic p, proving $\mathrm{Tr}_{S/R}(S) \subseteq pR$.

Modulo x, one has $S/xS = A[[t]]/(u_0 t^p) = A[[t]]/(t^p)$. Multiplication by an element of tS/xS strictly raises t-order, so its matrix has zero diagonal and $\mathrm{Tr}(tS) \subseteq xR$. Combining this with the first inclusion gives $\mathrm{Tr}(tS) \subseteq pR \cap xR$. Since R/pR is x-torsion-free, $pR \cap xR = pxR$.

(c). The trace extends $R[1/x]$-linearly to $S[1/t] = S \otimes_R R[1/x]$. Part (b) gives divisibility by p after localization, so U preserves $R[1/x]$. If $g \in R$, then

$$t^{-1}S = a^{\#}(x)^{-1} u_0 t^{p-1} S \subseteq x^{-1} t S,$$

and the second inclusion of (b) gives $\mathrm{Tr}(wt^{-1} b^{\#}(g)) \in pR$. Thus $U(g) \in R$.

(d). For $g \in R$,

$$wt^{-1} b^{\#}(x^{-m} g) \in t^{-(m+1)} S.$$

Write $m + 1 = p\kappa - j$ with $\kappa = \lceil (m + 1)/p \rceil$ and $0 \leq j \leq p - 1$. Since $x^\kappa = u_0^\kappa t^{p\kappa}$ up to units,

$$t^{-(m+1)} S = x^{-\kappa} t^j S.$$

If $j = 0$, use $\mathrm{Tr}(S) \subseteq pR$; if $j > 0$, use $\mathrm{Tr}(t^j S) \subseteq \mathrm{Tr}(tS) \subseteq pxR$. This gives precisely the two stated pole bounds, and both are strictly smaller than m for $p \geq 3$ and $m \geq 1$.

(e). Suppose $p \mid m+1$ and put $\kappa = (m+1)/p$ and $\widetilde{w} = wu_1^{-m}u_0^{\kappa}$. Then

$$U(x^{-m}) = x^{-\kappa}\frac{1}{p}\operatorname{Tr}(\widetilde{w}).$$

Modulo x, multiplication by $\widetilde{w}$ on $A[[t]]/(t^p)$ has all diagonal entries equal to $\widetilde{w}(0)$, so

$$\operatorname{Tr}(\widetilde{w}) = p\widetilde{w}(0) + xg$$

for some $g \in R$. Reducing modulo p and using part (b) gives $x\overline{g} = 0$ in $(A/pA)[[x]]$, hence $g \in pR$. Dividing by p therefore gives the claimed congruence. If w is a unit, so is $\widetilde{w}$, and its constant term is a unit of A. If $p \nmid m+1$, part (d) already places the image in the next lower pole step.

(f). The module $x^{-a}R/R$ carries its finite pole filtration. Part (d) sends each step into the next lower one, so $U^a = 0$. The same argument applies to stable subquotients and to every finite filtered module with the stated graded bounds. All sufficiently high positive powers are therefore zero. □

Theorem 6.2 (Matrix-valued Cartier trace)**.** *Retain the hypotheses of Theorem 6.1. Let $d \geq 1$ and $W \in M_d(S)$, and define*

$$U_W : R[1/x]^d \longrightarrow R[1/x]^d, \qquad U_W(\mathbf{g}) = \frac{1}{p}\operatorname{Tr}^{(d)}_{S/R}\big(t^{-1}Wb^{\#}(\mathbf{g})\big),$$

where trace is applied entrywise. Then U_W has the same integrality and strict pole bounds as the scalar operator. If $p \mid m+1$ and $\kappa = (m+1)/p$, its map on the resonant pole-graded piece is multiplication by

$$\big(u_1^{-m}u_0^{\kappa}W\big)(0) \in M_d(A).$$

If $W \in \mathrm{GL}_d(S)$, this matrix lies in $\mathrm{GL}_d(A)$.

Proof. Each output component is a finite sum of scalar operators of Theorem 6.1. Integrality and pole bounds therefore hold entrywise. On a resonant graded step, normalized trace followed by passage to the associated graded is evaluation at $t = 0$, giving the displayed matrix. If W is invertible, its determinant and hence the determinant of the evaluated matrix are units. □

Corollary 6.3 (Frame independence)**.** *Strict pole contraction is invariant under integral changes of source and target frames. For an endomorphism, nilpotence is invariant under an integral change of a common frame.*

Proof. Matrices in $\mathrm{GL}_d(R)$ preserve every lattice $x^{-m}R^d$ and each term of its pole filtration. Independent frame changes preserve the pole bounds; a common frame change conjugates the operator, and conjugation preserves nilpotence. □

6.1. **The congruence form of the engine.** For the branch application it is useful to record a mod-p variant whose input is exactly the shape produced by a partial Frobenius: a coefficient that is a Frobenius pullback modulo p.

Lemma 6.4 (Pure-model trace formula)**.** *Let $R_0 = \mathcal{O}[[x]][x^{-1}]$ and $S_0 = \mathcal{O}[[t]][t^{-1}]$ with $a^\#(x) = t^p$ and $b^\#(x) = t$. Then $1, t, \dots, t^{p-1}$ is an R_0-basis of S_0 and, for every integer q,*

$$\mathrm{Tr}_{S_0/R_0}(t^q) = \begin{cases} 0, & p \nmid q, \\ p\, x^{q/p}, & p \mid q. \end{cases}$$

In particular $\mathrm{Tr}_{S_0/R_0}(S_0) \subseteq pR_0$.

Proof. Writing $q = ap + r$ with $0 \le r < p$, multiplication by $t^q = x^a t^r$ permutes the basis up to x-factors, with zero diagonal when $r \neq 0$ and scalar diagonal x^a when $r = 0$. □

Proposition 6.5 (Frobenius-unit Cartier congruence)**.** *In the situation of Lemma 6.4, let $W = a^\#(w_0) + pW_1$ with $w_0 \in \mathcal{O}[[x]]$ and*

$$W_1 \in \mathcal{O}[[t]] \qquad (\textit{integral, not Laurent}),$$

and define $U_W(g) = \frac{1}{p}\,\mathrm{Tr}_{S_0/R_0}(t^{-1} W b^\#(g))$. Then $U_W(\mathcal{O}[[x]]) \subseteq \mathcal{O}[[x]]$, $U_W(1) \in p\,\mathcal{O}[[x]]$, and for $m \ge 1$,

$$U_W(x^{-m}) \equiv \begin{cases} 0, & p \nmid m+1, \\ w_0\, x^{-(m+1)/p}, & p \mid m+1 \end{cases} \quad (\mathrm{mod}\ pR_0).$$

If $w_0 \in \mathcal{O}[[x]]^\times$, every nonzero induced map of pole-graded pieces strictly lowers the pole order. The assertions hold entrywise for matrix-valued W and are preserved by any integral change of frame after which the coefficient again has the displayed form. If the geometry produces a Laurent error term $W_1 \in S_0$, the conclusions over $\mathcal{O}[[x]]$ must instead be imposed as the explicit lattice hypothesis $\mathrm{Tr}_{S_0/R_0}\big(t^{-1} W_1 b^\#(\mathcal{O}[[x]])\big) \subseteq p\,\mathcal{O}[[x]]$.

Proof. Error term. For $g \in \mathcal{O}[[x]]$ one has $b^\#(g) = g(t) \in \mathcal{O}[[t]]$, so $t^{-1} W_1 b^\#(g)$ is a Laurent series with exponents at least -1. By Lemma 6.4 only exponents divisible by p contribute to the trace, and since $p \ge 3$ no negative multiple of p is ≥ -1; each contributing exponent $q \ge 0$ yields $p\, x^{q/p}$ with $q/p \ge 0$. Hence

$$\mathrm{Tr}\big(t^{-1} W_1 b^\#(g)\big) \in p\,\mathcal{O}[[x]],$$

and the error contribution to $U_W(g)$ lies in $p\,\mathcal{O}[[x]]$; note that this containment genuinely uses $W_1 \in \mathcal{O}[[t]]$ and fails for Laurent W_1 (Remark 6.6).

Main term. Since $a^\#(w_0)$ acts R_0-linearly through the trace, on $g = x^{-m}$,

$$\frac{1}{p}\,\mathrm{Tr}\big(a^\#(w_0) t^{-m-1}\big) = w_0 \cdot \frac{1}{p}\,\mathrm{Tr}(t^{-m-1}),$$

which is 0 unless $p \mid m+1$ and equals $w_0 x^{-(m+1)/p}$ in the resonant case. For $g \in \mathcal{O}[[x]]$ the main term is $w_0 \cdot \frac{1}{p}\,\mathrm{Tr}(t^{-1} b^\#(g))$, and the same exponent count as for the error term places it in $\mathcal{O}[[x]]$; for $g = 1$ the only candidate exponent is -1, not divisible by p, so the main term vanishes and $U_W(1)$ reduces to the error contribution, in $p\,\mathcal{O}[[x]]$. Together with the error estimate this proves integrality on $\mathcal{O}[[x]]$ and the displayed congruences modulo pR_0. Since $(m+1)/p < m$ for $m \geq 1$, $p \geq 3$, resonance strictly contracts poles when w_0 is a unit. The matrix and frame statements follow entrywise and by recomputation; under the Laurent alternative, the stated lattice hypothesis replaces the error estimate verbatim. □

Remark 6.6 (The integral hypothesis on the error term is necessary). With a Laurent error term the integrality conclusions fail. Take $w_0 = 1$ and $W_1 = t^{1-p} \in S_0$. Then, for $g = 1$,

$$U_W(1) = \frac{1}{p}\operatorname{Tr}\bigl(t^{-1}(1 + p\,t^{1-p})\bigr) = \frac{1}{p}\operatorname{Tr}(t^{-1}) + \operatorname{Tr}(t^{-p}) = 0 + p\,x^{-1},$$

by Lemma 6.4; and $p\,x^{-1}$ lies in pR_0 but not in $\mathcal{O}[[x]]$. Thus a Laurent tail can push the operator out of the integral lattice even though every trace remains divisible by p, which is why the proposition requires $W_1 \in \mathcal{O}[[t]]$ or the explicit lattice hypothesis. Appendix A includes this example as a deterministic regression test.

Remark 6.7 (Role of the hypotheses). Condition (6.1) and the congruence $W \equiv a^{\#}(W_0) \pmod{p\,\mathcal{O}[[t]]}$ are not coordinate cosmetics. They are the inputs that identify the mod-p base with $(A/pA)[[t^p]]$ and force trace divisibility. Theorem 6.1 tolerates arbitrary unit distortions and tangential variables with *integral* pole bounds; Proposition 6.5 asks only for mod-p data with an integral error term and returns mod-p bounds, which Theorems 5.11 and 5.12 then convert into ordinary vanishing. Either route suffices for the branch application. Appendix A records finite-length checks in which the asserted congruences hold in all tested cases satisfying (6.1) and generally fail when it is violated; the computation is not used in any proof.

7. The defect-one patching endpoint

Counterexample 7.1 (Amplitude alone is insufficient)**.** *Let S be a regular local domain and $0 \neq x \in \mathfrak{m}_S$. The perfect complex $[S \xrightarrow{x} S]$ in degrees $0, 1$ has amplitude $[0, 1]$, but its top cohomology S/xS is not free over S.*

Proof. The complex is bounded and finite free. Its cohomology is $H^0 = 0$ and $H^1 = S/xS \neq 0$. A nonzero free module over the domain S cannot be annihilated by x, so S/xS is not free. □

Theorem 7.2 (Automatic depth in defect one)**.** *Let S be a regular complete local noetherian ring of dimension d, and let*

$$C = [P^0 \xrightarrow{d_C} P^1]$$

be a two-term complex of finite free S-modules. Let R be a complete local noetherian S-algebra acting on $H^0(C) \oplus H^1(C)$ so that the image of S acts by the given S-module structure. Assume

$$\dim R = d - 1, \qquad M := H^1(C) \neq 0.$$

Then:

(i) $H^0(C)=0$, $\operatorname{pd}_S M=1$, *and* $\operatorname{depth}_S M=d-1$*;*
(ii) *if* $\overline{R}$ *is the image of* R *in* $\operatorname{End}_S(H^0(C)\oplus H^1(C))$*, then* $\dim\overline{R}=d-1$ *and* M *is maximal Cohen–Macaulay over* $\overline{R}$*;*
(iii) *if* R *is regular, then* $R\to\overline{R}$ *is an isomorphism and* M *is finite free over* R*;*
(iv) *if, in the situation of* (iii)*, a quotient* $R\twoheadrightarrow T$ *acts faithfully on* M*, then* $R\cong T$.

Proof. Put $H=H^0(C)\oplus H^1(C)$. The image $\overline{R}\subseteq\operatorname{End}_S(H)$ is finite over $S/\operatorname{Ann}_S(H)$ and is a quotient of R. Hence

$$\dim S/\operatorname{Ann}_S(H)=\dim\overline{R}\le\dim R=d-1<d.$$

The regular local ring S is a domain, so $\operatorname{Ann}_S(H)$ contains a nonzero element s. The module $H^0(C)=\ker(d_C)$ is a submodule of the free module P^0 and is torsion-free. Since s annihilates it, $H^0(C)=0$. Thus

$$0\longrightarrow P^0\longrightarrow P^1\longrightarrow M\longrightarrow 0$$

is a free resolution. It has length exactly one: if M were projective, it would be free and hence torsion-free, contradicting $sM=0$ and $M\neq 0$. Auslander–Buchsbaum gives $\operatorname{depth}_S M=d-1$.

Let $\overline{S}=S/\operatorname{Ann}_S(M)$. The finite local extension $\overline{S}\to\overline{R}$ has the same radical of the maximal ideal on M, so depth computed by local cohomology agrees over the two rings. Hence

$$\operatorname{depth}_{\overline{R}}M=\operatorname{depth}_S M=d-1.$$

The inequalities

$$d-1\le\dim_{\overline{R}}\operatorname{Supp}M\le\dim\overline{R}\le\dim R=d-1$$

are therefore equalities, proving (ii).

If R is regular, it is a domain. The kernel of $R\twoheadrightarrow\overline{R}$ has height zero because the quotient has the same dimension as R, hence the kernel is zero. Auslander–Buchsbaum over R now gives $\operatorname{pd}_R M=0$, so M is finite free. Finally, the kernel of $R\twoheadrightarrow T$ annihilates the faithful T-module M and is therefore zero. □

Proposition 7.3 (Balanced variant)**.** *Let* S *be as above and let* $C=[S^r\xrightarrow{M_0}S^r]$*. Suppose a complete local noetherian ring* R *acts on* $M_1=H^1(C)$*, the image of* R *contains the image of* S*, and* $\dim R\le d-1$ *with* $M_1\neq 0$*. Then* $H^0(C)=0$*,* $\operatorname{pd}_S M_1=1$*,* $\operatorname{depth}_S M_1=d-1$*, and in fact* $\dim R=d-1$*. If* R *is regular, its action on* M_1 *is faithful and* M_1 *is finite free over* R*.*

Proof. Let $\overline{R}$ be the action image. As in the preceding proof, $\operatorname{Ann}_S(M_1)$ contains $0\neq s\in S$. Since $s\operatorname{coker}(M_0)=0$, the square matrix M_0 becomes surjective, hence bijective, over $\operatorname{Frac}(S)$. Thus its kernel is a torsion submodule of a free module and is zero. The remaining depth and dimension arguments are identical to those in Theorem 7.2. □

Remark 7.4. The balanced variant acts on the *top* cohomology only, with $H^0(C) = 0$ derived rather than assumed. This matters because cokernels commute with truncation and base change while kernels do not; an action on H^1 is what cohomology functors actually produce.

7.1. **Two-term pigeonhole patching.** The next theorems make every patched-level statement of a defect-one two-term patching argument unconditional: from finite-level data alone, the patched complex, the patched action, and the augmentation identification *exist*.

Lemma 7.5 (Minimal models). *Over a noetherian local ring $(\Lambda, \mathfrak{m}, k)$, every two-term complex $C = [\Lambda^{r_0} \xrightarrow{M} \Lambda^{r_1}]$ of finite free modules is isomorphic, after independent basis changes in source and target, to $[\Lambda^\kappa \xrightarrow{\mathrm{id}} \Lambda^\kappa] \oplus [\Lambda^{r_0-\kappa} \xrightarrow{M'} \Lambda^{r_1-\kappa}]$ with M' minimal (entries in $\mathfrak{m}$). The identity summand is split acyclic, so C is homotopy equivalent to a minimal complex, of uniquely determined ranks $r_1 - \kappa = \dim_k(H^1(C) \otimes_\Lambda k)$ and $r_0 - \kappa = (r_1 - \kappa) - (r_1 - r_0)$.*

Proof. If some entry of M is a unit, row and column operations move it to the $(1,1)$ position and clear its row and column, splitting off a factor $[\Lambda \xrightarrow{1} \Lambda]$, which is contractible. Iterating, at most $\min(r_0, r_1)$ times, leaves a block with all entries in $\mathfrak{m}$. The number κ of split steps equals $\mathrm{rank}_k(M \otimes k)$, which is invariant under basis changes; since cokernels commute with base change, $H^1(C) \otimes k = \mathrm{coker}(M \otimes k)$ has dimension $r_1 - \kappa$, and the Euler characteristic determines the remaining rank. ☐

Lemma 7.6 (Minimal quasi-isomorphisms are isomorphisms). *A quasi-isomorphism $f : C \to D$ of minimal two-term complexes of finite free modules over a noetherian local ring is an isomorphism of complexes.*

Proof. A quasi-isomorphism between bounded complexes of projectives is a homotopy equivalence; choose g and homotopies with $fg \simeq \mathrm{id}$, $gf \simeq \mathrm{id}$. For two-term complexes a homotopy is a single map $h : D^1 \to D^0$, and the homotopy identities read $(fg)^0 = \mathrm{id} + hM_D$, $(fg)^1 = \mathrm{id} + M_D h$, and similarly for gf. Minimality places the entries of M_C, M_D in $\mathfrak{m}$, so all four composites are congruent to the identity modulo $\mathfrak{m}$, hence invertible. In each degree f has invertible left and right composites and is therefore invertible. ☐

Lemma 7.7 (Deep congruence ideals). *Let $S_\infty = \mathcal{O}[[s_1, \dots, s_q]]$ with maximal ideal $\mathfrak{m} = (\varpi, s_1, \dots, s_q)$, and let $J_N = \big((1 + s_i)^{p^N} - 1 : \ \ i \le q\big)$ and $S_N = S_\infty / J_N$. Then $J_N \subseteq \mathfrak{m}^{N+1}$; consequently, for $N \ge n$ the canonical map $S_\infty/\mathfrak{m}^n \to S_N/\mathfrak{m}^n S_N$ is an isomorphism.*

Proof. Expand $(1+s)^{p^N} - 1 = \sum_{k \ge 1} \binom{p^N}{k} s^k$. By Kummer's theorem, $v_p\binom{p^N}{k} = N - v_p(k)$, so the k-th term lies in $\mathfrak{m}^{N - v_p(k) + k}$. Writing $k = p^j u$ with $p \nmid u$, one has $k \ge p^j \ge j + 1$, whence $N - j + k \ge N + 1$. Thus every term lies in $\mathfrak{m}^{N+1}$. The last statement follows because $J_N + \mathfrak{m}^n = \mathfrak{m}^n$ for $N \ge n$. ☐

Definition 7.8 (Two-term patching datum)**.** Fix a complete discrete valuation ring $\mathcal{O}$ with uniformizer ϖ and *finite* residue field $\mathbb{F}$; the diamond algebras $S_\infty = \mathcal{O}[[s_1, \ldots, s_q]]$, $\mathfrak{a} = (s_1, \ldots, s_q)$, and $S_N = S_\infty/J_N$ of Lemma 7.7; a candidate ring R^{c}, complete local noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, presented as a quotient of $\mathcal{O}[[x_1, \ldots, x_m]]$ by an ideal with a fixed finite generating set; and a fixed minimal base complex $D_0 = [\mathcal{O}^{r_0} \xrightarrow{M_0} \mathcal{O}^{r_1}]$ with a continuous action $a_0 : R^{\mathrm{c}} \to \mathrm{End}_{\mathcal{O}}(H^1(D_0))$ whose generator images act nilpotently on $H^1(D_0) \otimes \mathbb{F}$. A *two-term patching datum* is an infinite set $\mathcal{N} \subseteq \mathbb{Z}_{\geq 1}$ together with, for each $N \in \mathcal{N}$:

(D1) a minimal complex $D_N = [S_N^{r_0} \xrightarrow{M_N} S_N^{r_1}]$;
(D2) a continuous action $a_N : R^{\mathrm{c}} \to \mathrm{End}_{S_N}(H^1(D_N))$ with residually nilpotent generator images and with the image of $S_N \to \mathrm{End}_{S_N}(H^1(D_N))$ contained in $a_N(R^{\mathrm{c}})$;
(D3) an isomorphism of complexes $\beta_N : D_N \otimes_{S_N} \mathcal{O} \xrightarrow{\sim} D_0$ with $H^1(\beta_N)$ intertwining the induced action with a_0.

Theorem 7.9 (Two-term pigeonhole patching)**.** *Given a two-term patching datum, patched data exist: a minimal complex $C_\infty = [S_\infty^{r_0} \xrightarrow{M_\infty} S_\infty^{r_1}]$, a continuous action $a_\infty : R^{\mathrm{c}} \to \mathrm{End}_{S_\infty}(M_\infty^1)$ on $M_\infty^1 := H^1(C_\infty)$ whose image contains the image of the structure map $S_\infty \to \mathrm{End}_{S_\infty}(M_\infty^1)$, and an isomorphism of complexes $\gamma : C_\infty \otimes_{S_\infty} \mathcal{O} \xrightarrow{\sim} D_0$ with $H^1(\gamma)$ intertwining the a_∞-induced action with a_0. Moreover there is an increasing sequence $N(1) < N(2) < \cdots$ in $\mathcal{N}$ such that for every n the truncation of $(M_\infty, a_\infty, \gamma)$ modulo $\mathfrak{m}^n$ (and ϖ^n for γ) coincides with that of $(M_{N(n)}, a_{N(n)}, \beta_{N(n)})$.*

Proof. Step 1: truncated data live in finite sets. Fix $n \geq 1$ and consider $N \in \mathcal{N}$ with $N \geq n$. By Lemma 7.7, $S_N/\mathfrak{m}^n S_N = S_\infty/\mathfrak{m}^n =: S(n)$, a finite ring because its $\mathfrak{m}$-graded pieces are finite-dimensional over the finite field $\mathbb{F}$. Cokernels commute with base change, so $H^1(D_N) \otimes_{S_N} S(n) = \mathrm{coker}(M_N \bmod \mathfrak{m}^n)$, a finite module presented by the matrix $M_N \bmod \mathfrak{m}^n$. The R^{c}-action is S_N-linear, hence induces $a_{N,n} : R^{\mathrm{c}} \to \mathrm{End}_{S(n)}(\mathrm{coker}(M_N \bmod \mathfrak{m}^n))$, determined by the finite tuple $(a_{N,n}(x_j))_{j \leq m}$ by continuity and topological generation; likewise $\beta_N \bmod \varpi^n$ is a pair of matrices over $\mathcal{O}/\varpi^n$. The n-truncation

$$\tau_n(N) := \Big(M_N \bmod \mathfrak{m}^n;\ \big(a_{N,n}(x_j)\big)_{j \leq m};\ \beta_N \bmod \varpi^n\Big)$$

therefore ranges over a finite set, and the defining conditions — minimality, vanishing of the action on the fixed generators of the presentation ideal, residual nilpotence, containment of the $S(n)$-image in the algebra generated by the $a_{N,n}(x_j)$, and the intertwining of $H^1(\beta_N)$ modulo ϖ^n (meaningful because $\mathfrak{m}^n \subseteq \mathfrak{a} + (\varpi^n)$) — are equalities and memberships among the entries of $\tau_n(N)$, inherited under reduction $\tau_n \mapsto \tau_{n-1}$.

Step 2: diagonal choice. Set $\mathcal{N}_0 = \mathcal{N}$. Given the infinite $\mathcal{N}_{n-1}$, the map $N \mapsto \tau_n(N)$ takes values in a finite set, so some fibre is infinite; choose such an infinite fibre $\mathcal{N}_n \subseteq \mathcal{N}_{n-1}$ with common value σ_n. The system $(\sigma_n)_n$ is reduction-compatible. Choose $N(n) \in \mathcal{N}_n$ with $N(n) > N(n-1)$.

Step 3: limit objects. The first components of the σ_n define $M_\infty \in \mathrm{Mat}(S_\infty)$ with entries in $\mathfrak{m}$ by completeness; put $C_\infty = [S_\infty^{r_0} \to S_\infty^{r_1}]$ and $M_\infty^1 = \mathrm{coker}(M_\infty)$. Since cokernels commute with base change, $M_\infty^1/\mathfrak{m}^n M_\infty^1 = \mathrm{coker}(M_\infty \bmod \mathfrak{m}^n)$ carries from σ_n a compatible system of generator images, and because M_∞^1 is a finite module over the complete noetherian local S_∞, hence $\mathfrak{m}$-adically complete and separated,

$$a_\infty(x_j) := \varprojlim_n a_{(n)}(x_j) \in \mathrm{End}_{S_\infty}(M_\infty^1)$$

is well defined. Residual nilpotence at level $n = 1$ forces $a_\infty(x_j)^{k_0}(M_\infty^1) \subseteq \mathfrak{m}M_\infty^1$ for some k_0; S_∞-linearity then gives $a_\infty(x_j)^{k_0 n}(M_\infty^1) \subseteq \mathfrak{m}^n M_\infty^1$, and the $a_\infty(x_j)$ commute because all their truncations do. Hence every formal power series in the x_j with $\mathcal{O}$-coefficients converges at $(a_\infty(x_j))_j$, defining a continuous $\mathcal{O}$-algebra map $\mathcal{O}[[x_1, \ldots, x_m]] \to \mathrm{End}_{S_\infty}(M_\infty^1)$; it kills the fixed generators of the presentation ideal because it does so modulo every $\mathfrak{m}^n$, so it factors through R^{c}, giving a_∞. For the containment: the action of each $\lambda \in S_\infty$ agrees modulo every $\mathfrak{m}^n$ with the action of some element of R^{c}, so it lies in the closure of $a_\infty(R^{\mathrm{c}})$; and $a_\infty(R^{\mathrm{c}})$ is closed, being the continuous image of the profinite R^{c} (complete local noetherian with finite residue field) in the Hausdorff limit topology of $\mathrm{End}_{S_\infty}(M_\infty^1) = \varprojlim_n \mathrm{End}(M_\infty^1/\mathfrak{m}^n)$.

Step 4: augmentation. The third components of the σ_n define a pair of matrices over $\mathcal{O}$ in the limit, i.e. a map of complexes $\gamma : C_\infty \otimes_{S_\infty} \mathcal{O} \to D_0$; the truncations match because $\mathfrak{m}^n \subseteq \mathfrak{a} + (\varpi^n)$. Each component of γ is a square matrix over $\mathcal{O}$ invertible modulo ϖ, hence invertible; so γ is an isomorphism of complexes. Since H^1 of a two-term complex commutes with the truncations, $H^1(\gamma)$ intertwines the actions modulo every ϖ^n, hence exactly. The final truncationwise statement is the construction itself. ∎

Proposition 7.10 (Weak data upgrade and rank rigidity). *In Definition 7.8, weaken* (D1) *to: D_N is any bounded complex of finite free S_N-modules concentrated in degrees $[0,1]$, with* (D2) *unchanged on $H^1(D_N)$; and weaken* (D3) *to: $\beta_N : D_N \otimes_{S_N} \mathcal{O} \to D_0$ is a quasi-isomorphism of complexes such that $H^1(\beta_N)$ intertwines the actions. Then:*

(a) *the minimal ranks of all D_N are equal, namely $r_1 = \dim_{\mathbb{F}} H^1(D_0 \otimes_{\mathcal{O}} \mathbb{F})$ and $r_0 = r_1 - \chi(D_0 \otimes_{\mathcal{O}} \mathbb{F})$;*

(b) *replacing each D_N by a minimal model and β_N by the induced augmentation produces a strict patching datum in the sense of Definition 7.8;*

(c) *patched data exist, by Theorem 7.9;*

(d) *the strict datum is balanced (equal ranks $r_0 = r_1$, as required by Proposition 7.3) if and only if $\chi(D_0 \otimes \mathbb{F}) = 0$, a condition on the base complex alone.*

Proof. (b) By Lemma 7.5, choose a homotopy equivalence $e_N : D_N^{\min} \to D_N$ with $D_N^{\min}$ minimal, and transport the action to $H^1(D_N^{\min})$ by conjugation with the isomorphism $H^1(e_N)$; continuity, residual nilpotence, and the containment clause are invariant under conjugation by a module isomorphism. The composite $\beta'_N : D_N^{\min} \otimes \mathcal{O} \to D_N \otimes \mathcal{O} \to D_0$ is a quasi-isomorphism, because additive functors preserve homotopy equivalences; its source is minimal over $\mathcal{O}$ (entries of the matrix land in (ϖ)) and its target is minimal by hypothesis, so Lemma 7.6 makes β'_N an isomorphism of complexes, with $H^1(\beta'_N)$ intertwining the transported action with a_0.

(a) Base-changing the augmentation to $\mathbb{F}$: a quasi-isomorphism of bounded complexes of finite free $\mathcal{O}$-modules is a homotopy equivalence and survives $\otimes\mathbb{F}$, so $H^*(D_N \otimes_{S_N} \mathbb{F}) \cong H^*(D_0 \otimes \mathbb{F})$. By the uniqueness in Lemma 7.5, $r_1(N) = \dim_{\mathbb{F}} H^1(D_N \otimes \mathbb{F}) = \dim_{\mathbb{F}} H^1(D_0 \otimes \mathbb{F})$ and $r_0(N) = r_1(N) - \chi(D_N \otimes \mathbb{F}) = r_1 - \chi(D_0 \otimes \mathbb{F})$, independent of N. (c) follows from (b), and (d) is immediate from (a). $\square$

Corollary 7.11 (Patched endpoint from finite-level data)**.** *Suppose a two-term patching datum (strict or weak) exists in which additionally:*

(i) *R^{c} is regular local of dimension* $\dim S_\infty - 1$*;*
(ii) *$H^1(D_0) \neq 0$ and $\chi(D_0 \otimes \mathbb{F}) = 0$;*
(iii) *the finite-level Hecke algebra T_0 is the image of a_0 in* $\mathrm{End}_{\mathcal{O}}(H^1(D_0))$*.*

Then $M^1_\infty \neq 0$; M^1_∞ is finite free over R^{c}; and the induced surjection $R^{\mathrm{c}}/\mathfrak{a}R^{\mathrm{c}} \twoheadrightarrow T_0$ is an isomorphism, where the S_∞-algebra structure $\theta : S_\infty \to R^{\mathrm{c}}$ is defined by $\theta(\lambda) = a_\infty^{-1}(\lambda\text{-action})$ using the containment clause.

Proof. By Theorem 7.9 (after Proposition 7.10 if the datum is weak), patched data $(C_\infty, a_\infty, \gamma)$ exist, and $M^1_\infty/\mathfrak{a}M^1_\infty \cong H^1(D_0) \neq 0$, so $M^1_\infty \neq 0$. Proposition 7.3 applies with $S = S_\infty$ ($\dim S_\infty = 1 + q =: d$), $R = R^{\mathrm{c}}$ of dimension $d-1$, the action a_∞, and the containment of the S_∞-image; with regularity, a_∞ is injective, $R^{\mathrm{c}} \cong a_\infty(R^{\mathrm{c}})$, and M^1_∞ is finite free nonzero over R^{c}. The isomorphism $a_\infty : R^{\mathrm{c}} \xrightarrow{\sim} a_\infty(R^{\mathrm{c}})$ together with the containment defines θ, so $\mathfrak{a}R^{\mathrm{c}} := \theta(\mathfrak{a})R^{\mathrm{c}}$ makes sense and $\mathfrak{a}M^1_\infty = \mathfrak{a}R^{\mathrm{c}} \cdot M^1_\infty$. Then $M^1_\infty/\mathfrak{a}M^1_\infty$ is finite free and nonzero over $R^{\mathrm{c}}/\mathfrak{a}R^{\mathrm{c}}$, hence faithful; the intertwining says its action factors through the surjection $R^{\mathrm{c}}/\mathfrak{a}R^{\mathrm{c}} \twoheadrightarrow T_0$, and a surjection through which a faithful module action factors is an isomorphism (Lemma 7.13). $\square$

Remark 7.12 (What the theorem does and does not construct). Theorem 7.9 is a *diagonal compactness theorem for two-term patching data*: from a compatible infinite family of finite-level complexes, actions, and augmentations it constructs the limit objects. It does not construct the arithmetic Taylor–Wiles datum itself. Regularity and the dimension count, the deformation action, local–global compatibility, weight-flatness, and nonvanishing remain substantive arithmetic inputs, and every statement of the form "the patched objects exist by theorem" in this paper is to be read as "from the finite-level datum of Definition 7.8 (or its weak form)".

Lemma 7.13 (Faithful transport)**.** *Let R be a ring, M a faithful R-module, and $\varphi : R \to A$ a ring surjection such that the R-action on M factors through an A-module structure via φ. Then φ is an isomorphism.*

Proof. For $r \in \ker\varphi$ and $m \in M$: $r \cdot m = \varphi(r) \cdot m = 0$. So $\ker\varphi \subseteq \operatorname{Ann}_R(M) = 0$. □

Proposition 7.14 (Patched descent)**.** *Let R_∞ be a ring, $\mathfrak{a} \subseteq R_\infty$ an ideal, and M_∞ a finite free nonzero R_∞-module. Given a ring isomorphism $R_\infty/\mathfrak{a}R_\infty \cong R_x$ and a ring surjection $R_x \twoheadrightarrow A$ such that the induced R_x-action on $M := M_\infty/\mathfrak{a}M_\infty$ factors through an A-module structure, the map $R_x \to A$ is an isomorphism.*

Proof. $M \cong R_x^\rho$ with $\rho \geq 1$ is finite free nonzero, hence faithful; apply Lemma 7.13. □

Criterion 7.15 (Non-tautological finite-level input)**.** *To apply the theorems of this section, a coherent Taylor–Wiles argument must construct the finite-level complexes, actions, and augmentations of Definition 7.8 (in weak form, by Proposition 7.10); prove the defect-one dimension identity and regularity; prove nonvanishing of $H^1(D_0)$ and the balance $\chi(D_0 \otimes \mathbb{F}) = 0$; and identify $R^{\mathrm{c}}/\mathfrak{a}R^{\mathrm{c}}$ with the desired deformation ring. Every patched-level statement — existence of the patched complex, action, and augmentation; depth; freeness; faithfulness; and the descent isomorphism — is then a theorem. A clause saying that descent "recovers an isomorphism" is the conclusion, not an input.*

7.2. **Reduced derived patching.** For field-valued automorphy, a strictly weaker mechanism suffices.

Theorem 7.16 (Reduced derived-patching criterion)**.** *Let R be a noetherian ring, let $C^\bullet$ be a perfect R-complex, and let*

$$\phi : R \longrightarrow \operatorname{End}_{D(R)}(C^\bullet)$$

be the canonical action, sending r to the multiplication endomorphism; write $\mathbb{T} = \operatorname{im}(\phi)$. If

$$C^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}) \neq 0 \quad \textit{for every minimal prime } \mathfrak{p} \subset R,$$

then $\ker\phi$ is nilpotent and $R_{\mathrm{red}} \xrightarrow{\sim} \mathbb{T}_{\mathrm{red}}$.

Proof. Let $r \in \ker\phi$ and let $\mathfrak{p}$ be a minimal prime with $r \notin \mathfrak{p}$. Then multiplication by r is an automorphism of the localized perfect complex $C^\bullet_{\mathfrak{p}}$, while $\phi(r) = 0$ says it is zero in $D(R_{\mathfrak{p}})$; hence the identity of $C^\bullet_{\mathfrak{p}}$ is zero, i.e. $C^\bullet_{\mathfrak{p}} \simeq 0$, contradicting the nonzero derived fibre at $\mathfrak{p}$. So r lies in every minimal prime, i.e. in the nilradical, and $\ker\phi$ is nilpotent. Since $\mathbb{T} = R/\ker\phi$ and $\sqrt{\ker\phi} = \sqrt{(0)}$, passing to reductions gives the isomorphism. □

Remark 7.17. Theorem 7.16 separates two goals that are often conflated: literal integral $R = \mathbb{T}$ (which needs the depth, freeness, and descent package above) and reduced pointwise automorphy (which needs only one nonzero derived fibre per minimal prime). The separation shortens the route to field-valued applications; see Proposition 10.3.

8. Tangent spaces, Selmer groups, and local rigidity

8.1. The tangent–Selmer dictionary.

Theorem 8.1 (Characteristic-zero tangent–Selmer dictionary)**.** *Let $\rho : G_{F,S} \to \mathrm{GL}_2(E)$ be absolutely irreducible. Let a characteristic-zero local deformation problem with fixed determinant and tangent local conditions $\mathcal{L}_v \subseteq H^1(F_v, \mathrm{ad}^0 \rho)$ be represented by a complete local noetherian E-algebra R with residue field E. Then the tangent space of R is*

$$H^1_{\mathcal{L}}(F, \mathrm{ad}^0 \rho).$$

If R is a complete local $\Lambda_x = E[[t]]$-algebra representing a variable-weight problem and $d\kappa$ is the derivative of the weight parameter, then

$$\mathrm{Hom}_E(\Omega_{R/\Lambda_x} \otimes_R E, E) \cong \ker\big(d\kappa : H^1_{\mathcal{L}^{\mathrm{var}}}(F, \mathrm{ad}^0 \rho) \to E\big).$$

Proof. A lift to $E[\varepsilon]/(\varepsilon^2)$ has the form $\rho_\varepsilon(g) = (1 + \varepsilon c(g))\rho(g)$. Multiplicativity is equivalent to

$$c(gh) = c(g) + \mathrm{Ad}(\rho(g))c(h),$$

so c is a continuous cocycle. Strict conjugacy by $1 + \varepsilon X$ changes c by a coboundary. Since $\det(1 + \varepsilon c) = 1 + \varepsilon \operatorname{tr}(c)$, fixed determinant is exactly the trace-zero condition. The specified local deformation conditions cut out the Selmer subspace.

For the relative statement, the universal property of differentials gives

$$\mathrm{Hom}_E(\Omega_{R/\Lambda_x} \otimes_R E, E) = \mathrm{Der}_{\Lambda_x}(R, E).$$

These are precisely first-order points for which the composite $\Lambda_x \to R \to E[\varepsilon]$ is the augmentation, i.e., first-order deformations with zero derivative of the weight coordinate. Under the first part of the dictionary this is the displayed kernel. $\square$

8.2. A balanced fixed-weight Selmer structure.

Assume now that F is real quadratic, that $p \geq 5$ splits as $p\mathcal{O}_F = \mathfrak{p}_1\mathfrak{p}_2$, and that $\rho_x : G_{F,S} \to \mathrm{GL}_2(E)$ is continuous, absolutely irreducible, and *totally odd*. Assume also that its Hodge–Tate multisets are $\{0, 0\}$ at $\mathfrak{p}_1$ and $\{0, k-1\}$ at $\mathfrak{p}_2$. At each $v = \mathfrak{p}_i$, suppose

$$\rho_x|_{G_{F_v}} \sim \begin{pmatrix} \psi_{i,1} & * \\ 0 & \psi_{i,2} \end{pmatrix}, \qquad \eta_i = \psi_{i,1}\psi_{i,2}^{-1} \notin \{1, \epsilon_{\mathrm{cyc}}, \epsilon_{\mathrm{cyc}}^{-1}\}.$$

Put $W = \mathrm{ad}^0 \rho_x$ and $W^* = W(1)$. Let $\mathfrak{b}_v^0 \subset W$ be the trace-zero Borel preserving the ordinary line, with

$$0 \longrightarrow E(\eta_i) \longrightarrow \mathfrak{b}_v^0 \longrightarrow E \longrightarrow 0.$$

Definition 8.2 (Fixed-weight ordinary Selmer structure)**.** At real places put $\mathcal{L}_v = 0$. At finite places away from p, choose a balanced minimal fixed-type condition with $\dim \mathcal{L}_v = h^0(F_v, W)$. At $v = \mathfrak{p}_i$, let $\mathcal{L}_v$ be the image in $H^1(F_v, W)$ of those classes in $H^1(F_v, \mathfrak{b}_v^0)$ whose diagonal projection to

$H^1(F_v, E)$ has zero weight derivative. The variable-weight structure $\mathcal{L}^{\mathrm{var}}$ frees that one diagonal weight coordinate at $\mathfrak{p}_1$.

Lemma 8.3 (Lattice-limit extraction of the Greenberg–Wiles identity)**.** *Let $T \subset W$ be a $G_{F,S}$-stable $\mathcal{O}$-lattice, and $T^* = \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O}(1)) \subset W^*$ its Tate dual. For each place $v \in S \cup S_\infty$, let $\mathcal{L}_{v,T} \subseteq H^1(F_v, T)$ be the saturated preimage of $\mathcal{L}_v$, let $\mathcal{L}_{v,n}$ be its image in $H^1(F_v, T/\varpi^n)$, and let the dual conditions on T^*/ϖ^n be the exact annihilators of the $\mathcal{L}_{v,n}$ under local Tate duality. Write $q = \#(\mathcal{O}/\varpi)$. Then:*

(i) *all groups $H^i(F_v, T)$ and $H^i_{\mathcal{L}_T}(F, T)$ are finite over $\mathcal{O}$, and for $n \to \infty$,*
$$\#H^1_{\mathcal{L}_n}(F, T/\varpi^n) = q^{\,n \dim_E H^1_{\mathcal{L}}(F,W)} \cdot O(1)$$
and
$$\#\mathcal{L}_{v,n} = q^{\,n \dim_E \mathcal{L}_v} \cdot O(1),$$
and likewise on the dual side, with all $O(1)$ factors bounded independently of n;

(ii) *Wiles's finite-level formula* [21, Proposition 1.6] *for T/ϖ^n with the conditions $\mathcal{L}_{v,n}$ holds verbatim, and taking $\log_q$, dividing by n, and letting $n \to \infty$ yields*
$$\dim H^1_{\mathcal{L}}(F, W) - \dim H^1_{\mathcal{L}^\perp}(F, W^*) = h^0(F, W) - h^0(F, W^*) + \sum_v \big(\dim \mathcal{L}_v - h^0(F_v, W)\big).$$

Proof. (i) Finiteness of local and S-ramified global Galois cohomology of finite $\mathcal{O}$-modules, together with the exact sequences $0 \to H^i(T)/\varpi^n \to H^i(T/\varpi^n) \to H^{i+1}(T)[\varpi^n] \to 0$ coming from $0 \to T \xrightarrow{\varpi^n} T \to T/\varpi^n \to 0$, gives $\#H^i(T/\varpi^n) = q^{nr_i} \cdot O(1)$, where $r_i = \mathrm{rank}_{\mathcal{O}} H^i(T) = \dim_E H^i(W)$; the last equality holds because continuous cochains of a compact group with finitely generated coefficients commute with inverting p. For the Selmer groups, the comparison maps between $H^1_{\mathcal{L}_n}(F, T/\varpi^n)$ and the image of $H^1_{\mathcal{L}_T}(F, T)$ have kernels bounded by H^0-terms and lattice torsion, and cokernels bounded by the torsion of the local $H^1(F_v, T)$ and by $H^2(T)[\varpi^n]$; saturation of the $\mathcal{L}_{v,T}$ makes all of these of order bounded independently of n, and the same holds on the dual side, where the exact-annihilator conditions differ from the saturated dual conditions by groups of bounded order.

(ii) Wiles's formula applies to each finite module T/ϖ^n with the chosen conditions and their exact annihilators. Every factor in the formula has the shape $q^{n\cdot(\text{an } E\text{-dimension})} \cdot O(1)$ by (i), so applying $\log_q$, dividing by n, and passing to the limit kills every bounded factor and leaves exactly the displayed identity of dimensions. ☐

Proposition 8.4 (Local dimensions and global balance)**.** *Under the preceding hypotheses, including total oddness:*

(i) *for $v = \mathfrak{p}_i$, $\dim \mathcal{L}_v - h^0(F_v, W) = 1$, both in the split and nonsplit cases;*

(ii) *$h^0(F,W) = h^0(F,W^*) = 0$;*

(iii) *$\dim H^1_{\mathcal{L}}(F,W) = \dim H^1_{\mathcal{L}^\perp}(F,W^*)$;*

(iv)
$$\dim H^1_{\mathcal{L}^{\mathrm{var}}}(F,W) = 1 + \dim H^1_{(\mathcal{L}^{\mathrm{var}})^\perp}(F,W^*),$$
and $H^1_{\mathcal{L}}(F,W)$ is the kernel of the weight derivative on the variable-weight Selmer group.

Proof. Fix $v = \mathfrak{p}_i$ and abbreviate $H^j(-) = H^j(F_v,-)$. Local duality and the Euler characteristic formula give
$$h^0(E(\eta_i)) = h^2(E(\eta_i)) = 0, \qquad h^1(E(\eta_i)) = 1,$$
while $h^0(E) = 1$, $h^2(E) = 0$, $h^1(E) = 2$, spanned by the unramified class and $\ell_v = \log_p \circ \epsilon_{\mathrm{cyc}}$. The long exact sequence of the Borel extension is
$$0 \to H^0(\mathfrak{b}_v^0) \to H^0(E) \xrightarrow{\delta^0} H^1(E(\eta_i)) \to H^1(\mathfrak{b}_v^0) \to H^1(E) \to 0.$$
Fixing the weight cuts the two-dimensional $H^1(F_v,E)$ down to its unramified line. If the local representation is nonsplit, δ^0 is injective, the fixed-weight Borel space has dimension one, and $h^0(F_v,W) = 0$. If it is split, $\delta^0 = 0$, the fixed-weight Borel space has dimension two, and $h^0(F_v,W) = 1$. The map to $H^1(F_v,W)$ is injective because $H^0(F_v, W/\mathfrak{b}_v^0) = H^0(F_v, E(\eta_i^{-1})) = 0$. This proves (i).

Part (ii) follows from Schur's lemma and from the incompatibility of $\rho_x \cong \rho_x \otimes \epsilon_{\mathrm{cyc}}$ with the Hodge–Tate weights at $\mathfrak{p}_1$: twisting shifts both τ_1-weights from $\{0,0\}$ to $\{1,1\}$. Lemma 8.3 gives
$$\dim H^1_{\mathcal{L}}(F,W) - \dim H^1_{\mathcal{L}^\perp}(F,W^*) = h^0(F,W) - h^0(F,W^*) + \sum_v \big(\dim \mathcal{L}_v - h^0(F_v,W)\big).$$
The finite places away from p contribute zero, and the two places above p contribute $1+1$ by (i). At a real place, $\mathcal{L}_v = H^1(F_v,W) = 0$ for $p > 2$, while total oddness makes $\rho_x(c_v) \sim \mathrm{diag}(1,-1)$, whose adjoint action fixes exactly the diagonal trace-zero line: $h^0(F_v,W) = 1$, so each real place contributes -1. (Without oddness the real contribution would be -3 per place and the balance would fail; oddness is automatic for the Galois representations attached to Hilbert modular forms.) The total is $0 + 2 - 2 = 0$, proving (iii). Freeing the weight coordinate at $\mathfrak{p}_1$ increases the local term by one — the composite $H^1(\mathfrak{b}_v^0) \to H^1(E)$ is surjective in both local cases, so the fixed-weight subspace has codimension exactly one — proving the dimension formula in (iv); the kernel assertion is the definition of the fixed-weight condition, using the injectivity from (i) to make the diagonal projection well defined. □

Remark 8.5. The balance formula is bookkeeping, not a vanishing theorem. It shows that fixed-weight Selmer vanishing is equivalent to dual Selmer vanishing, but it does not prove either one.

8.3. **Cotangent rigidity and finite Frobenius detection.**

Theorem 8.6 (Finite-flat cotangent rigidity)**.** *Let $(\Lambda, \mathfrak{m}, k) \to (A, \mathfrak{n}, k)$ be a finite flat local map of noetherian local rings inducing the identity on residue fields. If $\Omega_{A/\Lambda} \otimes_A k = 0$, then $A \cong \Lambda$.*

Proof. The finite A-module $\Omega_{A/\Lambda}$ has zero residue fibre and is zero by Nakayama. The finite flat finitely presented map is therefore finite étale. Its special fibre is a finite étale local k-algebra with residue field k, hence is k. Thus A is finite free of rank one over Λ. The unit map is an isomorphism modulo the maximal ideal and hence an isomorphism by Nakayama. ☐

Lemma 8.7 (Derivations of a matrix algebra are inner)**.** *Every E-derivation of $M_n(E)$ is inner.*

Proof. Let e_{ij} be the standard matrix units and put

$$s = \frac{1}{n} \sum_{i,j} e_{ij} \otimes e_{ji}.$$

Then $\sum s^{(1)} s^{(2)} = 1$ and $(a \otimes 1)s = s(1 \otimes a)$ for every $a \in M_n(E)$. If d is a derivation, set $X = \sum s^{(1)} d(s^{(2)})$. Applying $\mu \circ (1 \otimes d)$ to the centralizing identity gives $aX = Xa + d(a)$, so $d(a) = aX - Xa$. ☐

Theorem 8.8 (Trace derivatives separate adjoint cohomology)**.** *Let $\rho : G \to \mathrm{GL}_n(E)$ be absolutely irreducible and let $c \in Z^1(G, \operatorname{ad} \rho)$. Put*

$$\dot{T}_c(g) = \operatorname{tr}(c(g)\rho(g)).$$

Then $\dot{T}_c(g) = 0$ for every $g \in G$ if and only if c is a coboundary.

Proof. A coboundary gives a commutator and hence zero trace. Conversely suppose all trace derivatives vanish. By Burnside's theorem, the span of $\rho(G)$ is $M_n(E)$. Define provisionally $d(\rho(g)) = c(g)\rho(g)$. If $\sum_i a_i \rho(g_i) = 0$, put $Y = \sum_i a_i c(g_i)\rho(g_i)$. For every $h \in G$, the cocycle identity and the assumed trace vanishing give $\operatorname{tr}(Y\rho(h)) = 0$. The trace pairing on $M_n(E)$ is nondegenerate and the $\rho(h)$ span the algebra, so $Y = 0$. Hence d is well defined and linear. The cocycle identity gives the Leibniz rule on the spanning elements, so d is a derivation. By Lemma 8.7, after replacing X by $-X$ if necessary, $d(a) = Xa - aX$. Therefore $c(g) = X - \rho(g)X\rho(g)^{-1}$. ☐

Remark 8.9. Theorem 8.8 is an infinitesimal form of the principle that an absolutely irreducible deformation is determined by its trace; compare Carayol's trace-determination theorem [6] and Mazur's deformation theory [15].

Corollary 8.10 (Finite Frobenius separator)**.** *Let $\rho : G_{F,S} \to \mathrm{GL}_n(E)$ be continuous and absolutely irreducible, and let $H \subseteq H^1(G_{F,S}, \operatorname{ad} \rho)$ be finite-dimensional. There is a finite set Q of places outside S, with $\#Q \le \dim_E H$, such that*

$$H \longrightarrow E^Q, \qquad [c] \longmapsto \big(\operatorname{tr}(c(\mathrm{Frob}_v)\rho(\mathrm{Frob}_v))\big)_{v \in Q}$$

is injective. Consequently, under the tangent–Selmer dictionary, a relative tangent space vanishes once the differentials of these finitely many unramified Hecke traces vanish.

Proof. The trace derivative $\dot{T}_c$ is a class function unchanged by coboundary shifts: $\rho_\varepsilon = (1+\varepsilon c)\rho$ is a genuine $E[\varepsilon]$-representation whose character is $\operatorname{tr}\rho(g) + \varepsilon \dot{T}_c(g)$, and characters are conjugation-invariant. It is continuous, since c and ρ are continuous and trace is polynomial.

Suppose $[c] \neq 0$. By Theorem 8.8 there is $g \in G_{F,S}$ with $\dot{T}_c(g) \neq 0$. Because the open normal subgroups $N \trianglelefteq G_{F,S}$ form a neighborhood basis of the identity and the target is nonarchimedean, continuity provides such an N with $\dot{T}_c$ nonvanishing on the entire coset gN; by the class-function property, $\dot{T}_c$ is then nonvanishing on the full conjugacy-stable set $\bigcup_h h\,gN\,h^{-1}$. The Chebotarev density theorem, applied to the finite quotient $G_{F,S}/N$ — the Galois group of a finite extension unramified outside S — supplies a place $v \notin S$ whose Frobenius class in $G_{F,S}/N$ meets gN; any Frobenius lift then lies in the conjugacy-stable set above, so $\dot{T}_c(\mathrm{Frob}_v) \neq 0$.

Thus the Frobenius functionals have common kernel zero on H. Choosing one functional nonzero on the current kernel and iterating, the dimension drops at each step, so at most $\dim_E H$ places are needed. The final assertion follows because the derivative of a Frobenius trace function is the displayed functional. □

8.4. **Seven local rigidity packages.**

Definition 8.11 (Kähler different)**.** For a finite type map $\Lambda \to A$, put

$$\mathfrak{D}^{\mathrm{K}}_{A/\Lambda} = \mathrm{Fitt}^0_A(\Omega_{A/\Lambda}).$$

Theorem 8.12 (Dense trace rigidity)**.** *Let $\Lambda \to A$ be finite flat and local, with A reduced and nonzero. Suppose A is generated as a Λ-algebra by $t_1, \dots, t_r$. If there are $\lambda_i \in \Lambda$ and a Zariski-dense set $\Sigma \subseteq \operatorname{Spec} A$ such that $t_i(x) = \lambda_i(x)$ for every $x \in \Sigma$, then $A \cong \Lambda$.*

Proof. Each $t_i - \lambda_i$ vanishes on a dense set and is therefore nilpotent. Reducedness gives $t_i = \lambda_i$, so $\Lambda \to A$ is surjective. A finite flat quotient of the local ring Λ has kernel generated by an idempotent; since $A \neq 0$, that idempotent is zero. □

Proposition 8.13 (Congruence-section criterion)**.** *Let Λ be noetherian local, let A be a finite flat local Λ-algebra, and suppose the structure map has a Λ-algebra section $s : A \twoheadrightarrow \Lambda$. Put $I = \ker(s)$ and*

$$\eta_s = s(\mathrm{Ann}_A(I)) \subseteq \Lambda.$$

Then

$$I/I^2 \cong \Omega_{A/\Lambda} \otimes_{A,s} \Lambda,$$

and the following are equivalent: $A \cong \Lambda$ via s; $I = 0$; $I/I^2 = 0$; and $\eta_s = \Lambda$.

Proof. The conormal sequence gives a surjection $I/I^2 \to \Omega_{A/\Lambda} \otimes_{A,s} \Lambda$. The map $a \mapsto a - \sigma(s(a)) \bmod I^2$, where $\sigma : \Lambda \to A$ is the structure map, is a Λ-derivation killing $\sigma(\Lambda)$ and constructs the inverse: on I it is the identity modulo I^2, and composing the two maps in the other order fixes $da \otimes 1$.

The implications $A \cong \Lambda \Rightarrow I = 0 \Rightarrow I/I^2 = 0$ are immediate. If $I = I^2$, the determinant trick supplies $e \in I$ with $(1-e)I = 0$; hence $1-e \in \mathrm{Ann}_A(I)$ and $s(1-e) = 1$, so $\eta_s = \Lambda$. Conversely, choose $e \in \mathrm{Ann}_A(I)$ with $s(e)$ a unit and rescale so that $s(e) = 1$. Then $e - 1 \in I$ and $e(e-1) = 0$, so e is an idempotent. Since A is local and $s(e) = 1$, one has $e = 1$ and $I = 0$. □

Lemma 8.14 (The base is closed)**.** *Let Λ be complete noetherian local and let A be a nonzero finite flat Λ-algebra. Then $\Lambda \to A$ is injective and its image is closed in the $\mathfrak{m}_\Lambda A$-adic topology.*

Proof. The finite flat nonzero Λ-module A is finite free of positive rank and hence faithful. The quotient A/Λ is finite over Λ; Krull intersection gives $\bigcap_n (\Lambda + \mathfrak{m}_\Lambda^n A) = \Lambda$, which is the closedness assertion. □

Theorem 8.15 (Exact Frobenius-trace rigidity)**.** *Let Λ be complete noetherian local, let A be a finite flat local Λ-algebra, and let $\rho_A : G_{F,S} \to \mathrm{GL}_2(A)$ be continuous. Give A its $\mathfrak{m}_\Lambda A$-adic topology, which by finiteness coincides with its $\mathfrak{m}_A$-adic topology, and call a subalgebra topologically generated by a set if it is the closure of the subalgebra that set generates. Suppose A is the closed Λ-subalgebra topologically generated by the values of the trace function $g \mapsto \mathrm{tr}\, \rho_A(g)$, and that*

$$\mathrm{tr}\, \rho_A(\mathrm{Frob}_v) \in \Lambda \qquad (v \notin S).$$

Then $A = \Lambda$. The same conclusion holds under the a priori weaker generation hypothesis that the unramified Frobenius traces alone topologically generate A over Λ; in that case the density step below is not even needed.

Proof. Unramified Frobenius conjugacy classes are dense in $G_{F,S}$ by Chebotarev, applied to the finite quotients cut out by open normal subgroups. The trace function is continuous and conjugation-invariant, and takes values in Λ on that dense set; since Lemma 8.14 makes Λ closed in A, it takes values in Λ everywhere. Under either generation hypothesis, A is the closure of a Λ-subalgebra generated by elements of the closed subring Λ, hence $A \subseteq \overline{\Lambda} = \Lambda$; the structure map is thus surjective, and it is injective by Lemma 8.14. □

Theorem 8.16 (Reduced special-fibre rigidity)**.** *Let $(\Lambda, \mathfrak{m}, k)$ be noetherian local and let A be a finite flat local Λ-algebra with residue field k. If $A/\mathfrak{m}A$ is reduced, then $A \cong \Lambda$. More generally, if $A/\mathfrak{m}A$ acts faithfully through a commutative semisimple k-algebra, the same conclusion holds.*

Proof. The finite-dimensional local k-algebra $A/\mathfrak{m}A$ is reduced, hence a field; its residue field is k, so it equals k. Since A is finite free over Λ, its rank is $\dim_k(A/\mathfrak{m}A) = 1$; the unit map is an isomorphism modulo $\mathfrak{m}$, hence surjective by Nakayama, and injective because a nonzero free module is faithful. For the second statement, a subalgebra of a commutative semisimple algebra — a finite product of fields — is reduced, so a faithful action through such an algebra makes $A/\mathfrak{m}A$ itself reduced. □

Proposition 8.17 (Seven rigidity routes). *Let $(\Lambda, \mathfrak{m}, k) \to (A, \mathfrak{n}, k)$ be finite flat and local with the same residue field, with A carrying the indicated Galois family when a route uses one. Each of the following seven conditions implies $A \cong \Lambda$:*

- (R1) $\Omega_{A/\Lambda} \otimes_A k = 0$*;*
- (R2) $\mathfrak{D}^{\mathrm{K}}_{A/\Lambda}$ *contains a unit;*
- (R3) *A is reduced, finitely generated by trace functions, and these agree with a Λ-valued reference family on a Zariski-dense subset;*
- (R4) *A has a local monogenic presentation $(\Lambda[X]/(P))_{\mathfrak{n}}$ whose residual root is simple;*
- (R5) *the structure map has a section with unit congruence ideal $\eta_s = \Lambda$;*
- (R6) *A is topologically trace-generated and every unramified Frobenius trace lies exactly in Λ;*
- (R7) *the fibre condition holds, in any of its three equivalent forms:*
  - (R7a) $\dim_k(A/\mathfrak{m}A) = 1$*;*
  - (R7b) *$A/\mathfrak{m}A$ is reduced;*
  - (R7c) *$A/\mathfrak{m}A$ acts faithfully through a commutative semisimple k-algebra.*

Proof. Route (R1) is Theorem 8.6. For (R2), a finitely presented module is zero exactly when its zeroth Fitting ideal is the unit ideal, so (R2) implies (R1). Route (R3) is Theorem 8.12. In (R4), the conormal sequence gives

$$\Omega_{A/\Lambda} \cong A\,dX/(P'(X)dX);$$

a simple residual root makes the cotangent fibre zero. Route (R5) is Proposition 8.13, and (R6) is Theorem 8.15. For (R7): a local artinian k-algebra with residue field k is reduced iff it is a field iff it equals k iff it has k-dimension one, and a subalgebra of a commutative semisimple algebra is reduced, so the three forms are equivalent (Theorem 8.16); a finite flat module over a local ring is finite free of rank $\dim_k(A/\mathfrak{m}A)$, so form (R7a) gives rank one, and the unit map is an isomorphism modulo $\mathfrak{m}$, hence an isomorphism. □

Proposition 8.18 (Exact infinitesimal form). *With Definition 8.2, fixed-weight Selmer vanishing is equivalent to: either $H^1_{\mathcal{L}^{\mathrm{var}}}(F, W) = 0$, or*

$$\dim_E H^1_{\mathcal{L}^{\mathrm{var}}}(F, W) = 1 \quad \textit{and} \quad d\kappa \neq 0 \textit{ on that line.}$$

In particular this pair of conditions is both sufficient and necessary.

Proof. By Proposition 8.4(iv), the fixed-weight group is the kernel of the linear functional $d\kappa : H^1_{\mathcal{L}^{\mathrm{var}}}(F, W) \to E$. A functional on a finite-dimensional space has zero kernel exactly when the space is zero, or the space is one-dimensional and the functional is nonzero: if the kernel is zero, then $\dim = \dim\ker + \dim\mathrm{im} \leq 1$. □

Proposition 8.19 (Finite ramification locus)**.** *Let Λ be a noetherian domain of dimension one, let A be finite flat over Λ, and assume the generic fibre is étale. Then $\operatorname{Supp}_A \Omega_{A/\Lambda}$ is a finite set of maximal ideals. At a maximal ideal outside this set, if the residue field agrees with that of its contraction to Λ, the completed local map is an isomorphism.*

Proof. The finite module $\Omega_{A/\Lambda}$ vanishes after tensoring with $\operatorname{Frac}(\Lambda)$, so some nonzero $d \in \Lambda$ annihilates it. Its support lies in $\operatorname{Spec}(A/dA)$, which is finite because A/dA is finite over the Artinian ring $\Lambda/(d)$. Outside the support, the completed local map is finite flat with zero differentials and equal residue fields; apply Theorem 8.6. $\square$

Counterexample 8.20 (Ramified degree two)**.** *Let $\Lambda = E[[T]]$ and $A = E[[z]]$ with $T \mapsto z^2$. Then A is finite flat, reduced, regular, and a complete intersection over Λ, but $\Omega_{A/\Lambda} \otimes_A E \neq 0$ and $A/TA \cong E[z]/(z^2)$ is nonreduced.*

Proof. Splitting power series into even and odd parts shows A is free of rank two over $\Lambda = E[[z^2]]$ with basis $\{1, z\}$, and $E[[z]]$ is a discrete valuation ring, hence regular and reduced. Completing the presentation $\Lambda[X]/(X^2 - T)$ at the maximal ideal exhibits A as the quotient of the regular ring $E[[T, X]]$ by the single regular element $X^2 - T$: a complete intersection. The conormal sequence gives $\Omega_{A/\Lambda} \cong A\,dz/(2z\,dz) \cong A/(z) \cong E \neq 0$ (characteristic zero makes 2 a unit), and $A/TA = E[z]/(z^2)$. Thus finite flatness, reducedness of A, regularity, and the complete-intersection property all hold while the cotangent fibre is nonzero and the special fibre is nonreduced: the weight map $T = z^2$ is ramified of degree two at the closed point. $\square$

Remark 8.21. Counterexample 8.20 shows that *classicality is not transversality*: a finite flat family, all of whose structural properties are as good as possible, can still have a nonreduced fixed-weight fibre at a repeated refinement. Any proof of Conjecture E must therefore invoke an arithmetic input distinguishing the chosen point from a ramification point; no purely formal proof exists.

8.5. **The local one-ring theorem.**

Theorem 8.22 (Local one-ring theorem)**.** *Suppose:*

(L1) *a support theorem has first shown that x belongs to the old support, so that $A_{\mathrm{old},x}$ and the natural chain of surjections*

$$\Lambda_x \longrightarrow R_x \twoheadrightarrow A_{\mathrm{full},x} \twoheadrightarrow A_{\mathrm{old},x}$$

of complete local E-algebras with residue field E are defined;

(L2) *a non-tautological patched comparison proves that the composite $R_x \to A_{\mathrm{old},x}$ is an isomorphism;*

(L3) *$A_{\mathrm{old},x}$ is finite flat over Λ_x;*

(L4) *at least one route of Proposition 8.17 holds for $\Lambda_x \to A_{\mathrm{old},x}$.*

Then every map in the displayed chain is an isomorphism. In particular the full-Iwahori local eigenvariety is smooth of dimension one at x, and its weight map is a local isomorphism.

Proof. By (L3)–(L4) and Proposition 8.17, the composite $\Lambda_x \to A_{\mathrm{old},x}$ is an isomorphism. By (L2) the composite of the two surjections $R_x \twoheadrightarrow A_{\mathrm{full},x} \twoheadrightarrow A_{\mathrm{old},x}$ is an isomorphism, so each surjection is injective, hence an isomorphism; in particular the full-versus-old identification $A_{\mathrm{full},x} \cong A_{\mathrm{old},x}$ is a *consequence* here, not a hypothesis. Finally $\Lambda_x \to R_x$ is a composition of isomorphisms. The geometric statement is the identification of $A_{\mathrm{full},x}$ with the regular one-dimensional $E[[t]]$. □

Remark 8.23. Earlier versions of this project imposed the branch identification $A_{\mathrm{full},x} \cong A_{\mathrm{old},x}$ as a separate hypothesis alongside (L2); as the proof shows, it is redundant. The genuine roles of the branch package (Conjecture C) are those in (L1): establishing the old support at x, identifying the old action, and proving the cohomological full/old comparison that feeds Proposition 9.7 — not an additional ring-theoretic input after (L2) is known.

9. Component support and integral faithfulness

Theorem 9.1 (Componentwise nonvanishing forces faithfulness)**.** *Let R be a reduced noetherian ring and let M be a finite R-module. If $M_{\mathfrak{p}} \neq 0$ for every minimal prime $\mathfrak{p}$ of R, then $\mathrm{Ann}_R(M) = 0$. Consequently, every quotient map $R \twoheadrightarrow T$ through which the action on M factors is an isomorphism.*

Proof. For a finite module, $\mathrm{Supp}_R(M) = V(\mathrm{Ann}_R(M))$. The hypothesis puts $\mathrm{Ann}_R(M)$ in every minimal prime, hence

$$\mathrm{Ann}_R(M) \subseteq \bigcap_{\mathfrak{p}\in \mathrm{Min}\, R} \mathfrak{p} = \sqrt{(0)} = 0.$$

The kernel of a quotient through which the action factors annihilates M and is therefore zero. □

Theorem 9.2 (Depth-support upgrade)**.** *Let R be noetherian local and let $0 \neq M$ be finite with $\mathrm{depth}_R M = \dim R$. Then every associated prime of M is a minimal prime of R of full dimension, and $\mathrm{Supp}_R(M)$ is a union of irreducible components. If a prime $\mathfrak{q}$ contains a unique minimal prime $\mathfrak{p}$, then*

$$\mathfrak{q} \in \mathrm{Supp}\, M \iff M_{\mathfrak{p}} \neq 0.$$

If moreover R is reduced and, for every minimal prime $\mathfrak{p}_j$, there is such a prime $\mathfrak{q}_j \in \mathrm{Supp}\, M$, then M is faithful.

Proof. For $\mathfrak{q} \in \mathrm{Ass}\, M$, Ischebeck's bound gives $\mathrm{depth}_R M \leq \dim R/\mathfrak{q}$. Since $\dim R/\mathfrak{q} + \mathrm{ht}\, \mathfrak{q} \leq \dim R$, the hypothesis forces $\mathrm{ht}\, \mathfrak{q} = 0$ and $\dim R/\mathfrak{q} = \dim R$. The minimal primes of $\mathrm{Supp}\, M$ belong to $\mathrm{Ass}\, M$, so $\mathrm{Supp}\, M$ is a union of components.

If $\mathfrak{q} \in \mathrm{Supp}\, M$, it lies over some associated, hence minimal, prime of R; uniqueness forces that prime to be $\mathfrak{p}$, so $\mathfrak{p} \in \mathrm{Ass}\, M \subseteq \mathrm{Supp}\, M$. The reverse implication holds because support is closed under specialization. The last assertion gives $M_{\mathfrak{p}_j} \neq 0$ for every minimal prime, so Theorem 9.1 applies. □

Theorem 9.3 (Maximal Cohen–Macaulay descent)**.** *Let R be a Cohen–Macaulay (for instance regular) local noetherian ring, let $M \neq 0$ be a finite free R-module, and let $a_1, \ldots, a_q \in \mathfrak{m}_R$ satisfy*

$$\dim R/(a_1, \ldots, a_q) = \dim R - q.$$

Then $(a_1, \ldots, a_q)$ is an R-regular sequence; $\overline{R} = R/(a_1, \ldots, a_q)$ is Cohen–Macaulay local of dimension $\dim R - q$; and $M/(a_1, \ldots, a_q)M$ is a nonzero finite free $\overline{R}$-module, in particular maximal Cohen–Macaulay over $\overline{R}$, so Theorem 9.2 applies to it.

Proof. In a Cohen–Macaulay local ring, elements of the maximal ideal whose quotient has dimension $\dim R - q$ form part of a system of parameters, and every part of a system of parameters is a regular sequence. A sequence regular on R is regular on the finite free module $M \cong R^\rho$, and the quotient $(R/(a))^\rho$ is again free. A Cohen–Macaulay local ring modulo a regular sequence is Cohen–Macaulay of dimension reduced by the length of the sequence, and a nonzero free module has depth equal to the depth of the ring, which now equals its dimension. □

Remark 9.4. Theorem 9.3 shows that maximal Cohen–Macaulayness of the descended patched module over the integral base is a formal consequence of the patched outputs of Section 7 — patched freeness over the regular patched ring plus the augmentation dimension count — and is therefore *not* an independent conjectural clause. This deletes the corresponding clause from Conjecture F.

Proposition 9.5 (Dense regular-weight support)**.** *Let Λ be a noetherian domain, let R be a finite flat Λ-algebra, and let M be a finite R-module. Suppose there is a Zariski-dense subset $\Sigma \subseteq \operatorname{Spec} \Lambda$ such that every point of $\operatorname{Spec} R$ lying above a point of Σ belongs to $\operatorname{Supp}_R(M)$. Then $M_\mathfrak{p} \neq 0$ for every minimal prime $\mathfrak{p}$ of R.*

Proof. Let $\mathfrak{p}$ be a minimal prime of R. First, $\mathfrak{p} \cap \Lambda = 0$: if $0 \neq \lambda \in \mathfrak{p} \cap \Lambda$, then λ is a nonzerodivisor on R (R is flat, hence torsion-free, over the domain Λ), but its image in the artinian local ring $R_\mathfrak{p}$ lies in the nilpotent maximal ideal, so $\lambda^n s = 0$ for some $s \notin \mathfrak{p}$, $s \neq 0$, a contradiction. Thus the component $V(\mathfrak{p}) = \operatorname{Spec}(R/\mathfrak{p})$ dominates $\operatorname{Spec} \Lambda$, and $\pi : V(\mathfrak{p}) \to \operatorname{Spec} \Lambda$ is finite, hence closed and surjective.

The fibre of π over the generic point (0) is a single point, namely $\mathfrak{p}$ itself: $R/\mathfrak{p}$ is a domain, finite and torsion-free over Λ, so $(R/\mathfrak{p}) \otimes_\Lambda \operatorname{Frac}(\Lambda)$ is a finite-dimensional domain over the field $\operatorname{Frac}(\Lambda)$, hence a field, and its spectrum is the one prime lying over (0).

Now consider the closed set $Z := \pi\big(\operatorname{Supp} M \cap V(\mathfrak{p})\big)$ in $\operatorname{Spec} \Lambda$ (finite maps are closed). Every $s \in \Sigma$ lies in Z: the fibre of π over s is nonempty by surjectivity, and every point of it lies above Σ, hence in $\operatorname{Supp} M$ by hypothesis.

So Z contains the dense set Σ and is closed, whence $Z = \operatorname{Spec}\Lambda \ni (0)$. Therefore some point of $\operatorname{Supp} M \cap V(\mathfrak{p})$ lies over (0); by the previous paragraph that point is $\mathfrak{p}$. Hence $\mathfrak{p} \in \operatorname{Supp} M$, i.e. $M_{\mathfrak{p}} \neq 0$. □

Counterexample 9.6 (Why component support is necessary)**.** *Let*

$$R = \mathcal{O}[[T]][Z]/(Z(Z-p)).$$

The ideals (Z) and $(Z-p)$ are the two minimal primes; R is reduced; the two components are disjoint on the generic fibre and meet at the closed point. A module supported only on the component (Z) can have the desired completed local ring at every characteristic-zero point of that component while vanishing at the other minimal prime, and $T \mapsto 0$, $Z \mapsto p$ is an integral lift of R living entirely on the second component. Thus a local one-ring theorem on one component does not imply an integral all-components theorem.

Proof. In $\mathcal{O}[[T]][Z]$ one has $(Z) \cap (Z-p) = (Z(Z-p))$: if $h = fZ = g(Z-p)$, then reducing modulo Z in the p-torsion-free ring $\mathcal{O}[[T]]$ gives $\overline{g} \cdot (-p) = 0$, so $Z \mid g$ and $h \in (Z(Z-p))$. Hence R embeds in $R/(Z) \times R/(Z-p) \cong \mathcal{O}[[T]] \times \mathcal{O}[[T]]$, a product of domains, so R is reduced with the two stated minimal primes, both contained in the maximal ideal (ϖ, T, Z) (note $Z - p \in (\varpi, T, Z)$). On the generic fibre $e = Z/p$ is idempotent, since $Z^2 = pZ$, and splits $R[1/p]$ into two copies of $\mathcal{O}[[T]][1/p]$; at every E-point of the first factor the completed local ring is $E[[T]] = \Lambda_x$. The module $R/(Z)$ is nonzero at (Z) and vanishes after localization at $(Z-p)$. Finally $T \mapsto 0$, $Z \mapsto p$ kills $Z(Z-p)$ and defines an $\mathcal{O}$-algebra map $R \to \mathcal{O}$ that does not factor through $R/(Z)$, since $Z \mapsto p \neq 0$. □

Proposition 9.7 (Support transfer)**.** *Let T be a commutative noetherian ring, $i : O^\bullet \to C^\bullet$ a T-equivariant map of complexes of T-modules with finite cohomology modules, $Q^\bullet = \operatorname{Cone}(i)$, and e an idempotent acting compatibly on all cohomology, commuting with the T-action and the maps. If $e\, H^n(Q^\bullet) = 0$ for all n, then for every n:*

- (i) *$e\, H^n(O^\bullet) \to e\, H^n(C^\bullet)$ is an isomorphism;*
- (ii) *$\operatorname{Supp}_T(eH^n(C^\bullet)) = \operatorname{Supp}_T(eH^n(O^\bullet))$;*
- (iii) *if $\mathfrak{m}_x \in \operatorname{Supp}_T(eH^n(C^\bullet))$, then the completed old local algebra at x is nonzero, legitimizing the symbol $A_{\mathrm{old},x}$.*

Proof. The cone triangle gives the long exact sequence $\cdots \to H^{n-1}(Q) \to H^n(O) \xrightarrow{\alpha} H^n(C) \to H^n(Q) \to \cdots$. Applying the exact idempotent e and using $eH^{n-1}(Q) = eH^n(Q) = 0$ makes $e\alpha$ an isomorphism, whence (i) and (ii). For (iii), the localization $(eH^n(O))_{\mathfrak{m}_x}$ is then nonzero and finite over the noetherian local ring $T_{\mathfrak{m}_x}$; by Nakayama and faithful flatness of completion its completion is nonzero, and the completed old local algebra acts on it with 1 acting as the identity, hence is a nonzero ring. □

10. The conjectural package and its evidence

The six remaining inputs fall naturally into three groups. Each is stated in the smallest form our theorems permit: every clause that earlier versions of this project carried and that is now a theorem has been deleted, with the deletion justified in the accompanying discussion. None of the statements below contains the conclusion it is later used to prove.

10.1. **Group I: the singular operator and comparison.**

Conjecture A (Singular partial Fontaine operator)**.** *On the ordinary infinite-level Hilbert modular surface there is a finite cover by Hodge–Tate charts U_i, with partial period coordinates z_i and frames of the source and target line bundles, such that the boundary map of the relevant square-zero period extension defines a global $G(\mathbb{Q}_p)$-equivariant first-order operator*

$$\mathcal{N}_{\tau_1} : \omega_{\mathrm{la}}^{(1,k)} \longrightarrow \omega_{\mathrm{la}}^{(-1,k)} \otimes \Omega^1_{\tau_1}(1).$$

Its principal symbol is the nonzero partial Kodaira–Spencer symbol, and in each chart it is a nonzero scalar multiple of ∂_{z_i}. The regular operator in the τ_2-direction is the correctly normalized order-$(k-1)$ BGG–Fontaine operator $D_{\tau_2,k}$.

Evidence and reduction. For modular curves, Pan constructs the corresponding infinite-level operators and obtains classicality consequences [16, 17]; Rodríguez Camargo's geometric Sen theory and Jiang's differential-operator formalism provide the general period-geometric setting [18, 14], and p-adic Maass–Shimura operators exist on μ-ordinary Igusa varieties [7]. The structural theorems of Section 3 pin down the shape of the conjecture: Proposition 3.3 and Proposition 3.4 prove that a first-order operator cannot exist classically, even through an auxiliary bundle, so the operator must be genuinely infinite-level; and Lemma 3.7 shows that its classical shadow is forced to be the Bol operator $\mathcal{B}_1$. Lemma 3.8 shows that on a chart the operator may be obtained as the difference of the étale and de Rham multiplicative sections of the square-zero period extension: it is then automatically a derivation, determined by its value on the single coordinate z_i. Lemma 3.6 reduces equality with the period boundary to two checks: equality of symbols and equality on one frame, the latter being the existence of one horizontal lift of the frame. Proposition 3.10 prices equivariance at one group cohomology class, which vanishes for free if the operator is the unique one satisfying any G-invariant normalization; and Proposition 3.11 reduces the Čech acyclicity needed for gluing to Banach stages, where the perfectoid acyclicity theorems [19] apply. The remaining content is geometric: produce the two sections on each chart, prove the coordinate difference is a nonzero constant, and construct one horizontal lift of the frame.

Conjecture B (Joint kernels)**.** *There are bounded Hecke-equivariant complexes $K^\bullet_{\mathrm{cl}}$ and $K^\bullet_{\mathrm{an}}$ and a chain map $\Phi : K^\bullet_{\mathrm{cl}} \to K^\bullet_{\mathrm{an}}$ such that:*

(B1) *after non-Eisenstein localization and projection to the chosen ordinary refinement, $K^\bullet_{\mathrm{an}}$ is the joint-kernel complex of $\mathcal{N}_{\tau_1}$ and $D_{\tau_2,k}$, and Φ is a quasi-isomorphism onto it;*

(B2) *the comparison is compatible with the transition maps of the chart cover, intersectionwise;*

(B3) *the compactified statement holds: the localized boundary cone is acyclic;*

(B4) *the identification is compatible with the unramified Hecke action at the finite places needed to compare eigensystems — an arithmetic interpolation clause, listed separately because it is of a different type from the sheaf-theoretic clauses* (B1)–(B3).

Evidence and reduction. Theorem 4.2 and Proposition 4.3 compute the local analytic kernels exactly once Conjecture A identifies the operators with partial derivatives, and Theorem 4.4 computes the rectangular kernel of the descended second-order pair, which Lemma 4.8 exhibits as a canonical self-extension of the first-order kernel; Remark 4.9 explains exactly how the two kernel theories interact, and identifying the classical layer inside the rectangle is part of this conjecture's content. Proposition 3.9 removes the global Čech diagram chase: it is enough to prove the comparison on every finite intersection and on the localized boundary. Two further proved reductions bracket the problem. Filtered: if both complexes carry finite compatible filtrations and the map on every graded piece is a regular-weight comparison, the total map is a quasi-isomorphism by Proposition 5.9. Concentrated: if the localized complexes have cohomology in one common degree, clauses (B1)–(B3) collapse to a single module isomorphism $H^d(\Phi)$, by Proposition 5.8. And the globalized rectangular criterion, Theorem 4.11, proves the descended second-order form of the comparison outright once the morphism, its intersectionwise behavior, Čech acyclicity, the symbol identification, horizontality, and boundary control are supplied. Jiang's Hilbert classicality theorem covers regular parallel weights [13]; the missing point is the singular nonparallel comparison and its boundary behavior.

10.2. Group II: branch separation and coherent patching.

Conjecture C (Derived Goren–Oort branch separation)**.** *Before completion at x, the normalized $U_{\mathfrak{p}_1}$ correspondence induces, through chain-level intertwining homotopies as in Corollary 5.5, an endomorphism of the two-degeneracy old cone attached to each finite Hasse-pole quotient. After a finite faithfully flat cover and normalization of the correspondence:*

(C1) *the separable contribution is null-homotopic on the old cone;*

(C2) *the partial-Frobenius contribution satisfies, on each chart, the completed local model of Theorem 6.1 with the unit hypothesis* (6.1)*, or the congruence model of Proposition 6.5 with invertible reduced coefficient;*

(C3) *the normalization-conductor contribution — the term controlled by the conductor sheaf $\mathcal{Q}_Z$ of Proposition 5.13 for the normalized correspondence — and the Frobenius contribution preserve the Hasse-pole* *filtration termwise and are null-homotopic on every associated graded complex modulo p;*

(C4) *all homotopies and filtrations are compatible with Čech differentials, deck transformations, Hecke localization, and faithfully flat descent;*

(C5) *every $U_{\mathfrak{p}_1}$-eigenvalue on the localized regular-weight new part has positive valuation.*

Proposition 10.1 (Derived support consequence)**.** *Assume Conjecture C. Then the ordinary parts of the singular and regular-weight old cones are acyclic; the localized ordinary old and full cohomologies agree Hecke-equivariantly; the point x lies on the old support if and only if it lies on the full support; and $A_{\mathrm{full},x} \to A_{\mathrm{old},x}$ is an isomorphism.*

Proof. By (C1), the separable term is null-homotopic, so the induced endomorphism of the cone is homotopic to the residual contribution, as in the proof of Theorem 5.14. By (C2)–(C4), Theorem 6.1 or Proposition 6.5 verifies, chart by chart and compatibly by (C4), that the residual contribution preserves the Hasse-pole filtration termwise and is null-homotopic on every graded piece modulo p — exactly the hypotheses of Theorem 5.12 (alternative (a) of Theorem 5.14), whose complexes here are bounded and finite free. Hence the powers of the normalized $U_{\mathfrak{p}_1}$ tend to zero p-adically on the cone's cohomology and every ordinary projector kills the singular old cone. For the regular-weight old cone, (C5) and Proposition 5.15 give ordinary acyclicity directly. Proposition 9.7 then identifies the ordinary old and full cohomologies Hecke-equivariantly, transfers supports, and legitimizes $A_{\mathrm{old},x}$; because the Hecke algebras are by definition the faithful images of the fixed algebra $\mathbb{H}$ on these modules (convention (S7) of §2.1), equality of the localized modules with intertwined actions gives equality of the completed local Hecke images: $A_{\mathrm{full},x} \cong A_{\mathrm{old},x}$. □

Evidence and reduction. The local half of the conjecture is a theorem in this paper: Theorem 6.1 proves the complete local calculation with arbitrary unit distortions and tangential variables, Theorem 6.2 handles coefficient bundles, and Proposition 6.5 shows that mod-p data of exactly the shape a partial Frobenius produces suffice. Theorem 5.6 and Theorem 5.11 show that a chain-level disjoint branch splitting is unnecessary: null-homotopy of the separable term and filtration lowering modulo p in the homotopy category suffice. Proposition 5.8 converts, after localization to a single cohomological degree, every homotopy obligation into a module identity: the intertwining homotopies of (C1) exist as soon as the classical U–degeneracy relations hold on cohomology, and the separable null-homotopy holds as soon as the separable branch kills the cone's single cohomology module. Clause (C5) is the classical positive-new-slope input, which Proposition 5.15 converts into ordinary acyclicity. Diamond's saving trace, the Goren–Oort geometry of Tian–Xiao, and the minimal-cone analysis of Diamond–Kassaei provide the closest global geometric inputs [8, 20, 9]. What is not yet supplied by those statements is the required endomorphism of the *two-degeneracy* old cone together with the conductor term and the descent compatibilities. The chart-local model has also been stress-tested numerically with a negative control (Appendix A); the checks are evidence of implementability, not proof.

Conjecture D (Finite-level coherent patching datum)**.** *For the one-variable, one-prime-Iwahori Hilbert-surface deformation problem at the singular specialization, there exists an infinite set of Taylor–Wiles levels and, at each, a weak two-term patching datum in the sense of Proposition 7.10 — perfect ordinary coherent complexes of amplitude* $[0,1]$ *over the diamond rings with continuous deformation actions on top cohomology commuting with the Hecke action, and augmentation quasi-isomorphisms to the fixed singular-weight base complex — such that in addition:*

- (D1) *the candidate ring* R^{c} *is the global deformation ring of Standing Hypothesis 2.1 with Taylor–Wiles auxiliary conditions, regular of dimension* $\dim S_\infty - 1$ *— the regularity and dimension count being the standard smoothness and tangent-space computation for the listed local deformation conditions together with the Taylor–Wiles prime count, carried out at the singular weight;*
- (D2) $H^1(D_0) \neq 0$ *and* $\chi(D_0 \otimes \mathbb{F}) = 0$*;*
- (D3) *writing* $\theta : S_\infty \to R^{\mathrm{c}}$ *for the structure map produced from the datum by Corollary 7.11, and* $\mathfrak{a}R^{\mathrm{c}} := \theta(\mathfrak{a})R^{\mathrm{c}}$*, the quotient* $R^{\mathrm{c}}/\mathfrak{a}R^{\mathrm{c}}$ *is the singular-weight ordinary deformation ring, compatibly with the factorization of the deformation action through the finite-level old Hecke image, whose completed generic-fibre localization at* x *is* $A_{\mathrm{old},x}$*;*
- (D4) *the descended module* $M_0 := M^1_\infty/\mathfrak{a}M^1_\infty \cong H^1(D_0)$ *(the identification being* $H^1(\gamma)$ *of Theorem 7.9) is weight-flat: finite flat over* Λ^{int} *before localization, hence* $A_{\mathrm{old},x}$ *finite flat over* Λ_x *after localization and completion at* x*.*

No patched-level statement appears among these clauses: by Theorem 7.9, Proposition 7.10, and Corollary 7.11, the patched complex, the patched action, the augmentation identification, patched freeness, and the descent isomorphism $R_x \cong A_{\mathrm{old},x}$ *are theorems once the finite-level data exist (Remark 7.12); note that the symbol* $\mathfrak{a}R^{\mathrm{c}}$ *in* (D3) *is well defined because* θ *is constructed by that corollary before the clause is imposed.*

Evidence and reduction. Earlier versions of this project, and of the surrounding literature, formulate coherent patching with patched-level existence clauses — a regular patched ring, a patched complex, an augmentation identification. Section 7 proves all of these from finite-level data, so the conjecture above carries none of them; this is the sharpest formal contraction in the paper. Grossi's higher Hida theory and the quadratic-Hilbert complexes of Grossi–Loeffler–Zerbes supply natural finite-level candidates for the complexes [11, 10]; we note that the January 2026 revision of [10] reworks its control and classicality results through spherical-level rather than

Iwahori-level geometry, so the one-prime-Iwahori complexes needed here must be adapted from that version rather than quoted, and we cite the revision explicitly. Calegari–Geraghty provide the defect-one framework and carry out the elliptic weight-one analogue of the whole mechanism [5]. The defect-one dimension identity has a proved characteristic-zero shadow: the Greenberg–Wiles balance of Proposition 8.4(iii) says the singular-weight local condition at $\mathfrak{p}_1$ contributes the same excess as a regular ordinary condition, so the standard Taylor–Wiles count goes through at the singular weight provided its integral analogue holds; the risk is localized to one integral local computation at $\mathfrak{p}_1$. Nonvanishing is a single ordinary class at base level. The remaining research content is concentrated at finite level: the singular specialization of the higher-Hida complexes, the deformation action (local–global compatibility at partial weight one), the augmentation comparison, the integral defect-one count, nonvanishing, and weight-flatness.

10.3. Group III: local transversality and integral support.

Conjecture E (Local transversality at nonexceptional points). *Suppose ρ_x has no self-twist: it is not induced from a quadratic extension of F, so x carries no real- or complex-multiplication structure. Then*

$$H^1_{\mathcal{L}}(F, \mathrm{ad}^0 \rho_x) = 0.$$

Equivalently, the relative cotangent fibre of the variable-weight ordinary local deformation ring over Λ_x vanishes. Equivalently, after Conjectures C and D, at least one of the seven routes of Proposition 8.17 holds for $\Lambda_x \to A_{\mathrm{old},x}$. For a point x not satisfying the self-twist hypothesis, the displayed vanishing is not *conjectured; it becomes a pointwise transversality hypothesis to be verified or refuted by the finite diagnostics below.*

Remark 10.2 (Why the scope restriction is necessary). The elliptic-modular analogue fails precisely at p-split RM points, where the eigencurve is ramified over weight space [3], and Counterexample 8.20 realizes the same failure shape abstractly. A version of the conjecture quantifying over all ordinary partial-weight-one points would therefore be false-adjacent; the self-twist exclusion names the known failure mechanism, and the paper's own counter-evidence is the reason the scope cannot remain implicit.

Evidence, limitations, and diagnostics. The eigencurve is smooth at regular classical weight-one points, with an explicit criterion for étaleness over weight space [2]. This supports the expectation that a suitable nonexceptional Hilbert point is transverse. It is not evidence for a universal theorem: real-multiplication points can be ramified, as analyzed by Betina [3], and Counterexample 8.20 shows that finite flatness, regularity of the total space, and expected dimension are formally insufficient; any proof must use an arithmetic input distinguishing x from a ramification point. Proposition 8.18 shows the conjecture is *exactly* equivalent to a one-dimensional variable-weight tangent space together with one nonzero weight derivative, so its remaining content is one dimension bound plus one nonvanishing number. Corollary 8.10 turns tangent vanishing into finitely many explicit Frobenius-trace derivative tests. Proposition 8.19 shows that, after generic étaleness, only finitely many points can fail; and the fibre route (R7) of Proposition 8.17, in its three equivalent forms, identifies the conjecture with a multiplicity-one statement — reducedness of the fixed-weight Hecke fibre — approachable by q-expansion or congruence-module methods [12]. These are strong diagnostics, not a proof of the conjecture at the chosen x.

Conjecture F (Integral component support and control)**.** *Let R^{int} be the reduced integral residual-local deformation ring and let $R^{\mathrm{int}} \twoheadrightarrow \mathbb{T}^{\mathrm{int}}$ be the finite-level deformation-to-Hecke surjection acting on the finite descended patched module M. The following hold:*

- (I1) *$M_{\mathfrak{p}} \neq 0$ for every minimal prime $\mathfrak{p}$ of R^{int};*
- (I2) *singular fixed-weight control identifies the relevant special fibre with classical weight-$(1, k)$ Hecke classes;*
- (I3) *local–global compatibility holds at the finite places needed to identify the prescribed deformation conditions.*

Evidence and reduction. Earlier formulations carried a fourth clause — maximal Cohen–Macaulayness of M over R^{int} — which Theorem 9.3 proves from the patched outputs of Conjecture D; it has accordingly been deleted. The algebraic endpoint of clause (I1) is formal: Theorem 9.1 shows that nonvanishing at every minimal prime makes the patched action faithful, and Theorem 9.2 shows that, M being maximal Cohen–Macaulay, its support is a union of irreducible components, so *one* well-located automorphic point on each component decides the entire component. Counterexample 9.6 shows the clause cannot be removed: a perfect local theorem on one generic component does not control an integral lift on another component. Finally, the following proposition replaces (I1), for the reduced conclusion, by a support condition located exactly where known modularity theorems live.

Proposition 10.3 (Reduced-support alternative)**.** *Assume the descended module M is finite over R^{int}, that R^{int} is reduced and finite flat over Λ^{int}, and that there is a Zariski-dense subset $\Sigma \subseteq \operatorname{Spec} \Lambda^{\mathrm{int}}$ of regular weights such that every point of $\operatorname{Spec} R^{\mathrm{int}}$ above Σ lies in $\operatorname{Supp} M$. Then clause* (I1) *of Conjecture F holds; consequently $\operatorname{Ann}_{R^{\mathrm{int}}}(M) = 0$ and $R^{\mathrm{int}} \cong \mathbb{T}^{\mathrm{int}}$. If instead of the pointwise hypothesis one has a perfect complex with nonzero derived fibre at every minimal prime, then at least $R^{\mathrm{int}}_{\mathrm{red}} \cong \mathbb{T}^{\mathrm{int}}_{\mathrm{red}}$ (Theorem 7.16).*

Proof. Proposition 9.5 applied to $\Lambda^{\mathrm{int}} \to R^{\mathrm{int}}$ and M gives $M_{\mathfrak{p}} \neq 0$ at every minimal prime, which is (I1); Theorem 9.1 concludes. □

10.4. **Status table and dependency order.**

Input	Proved reduction	Remaining mathematical construction
A	Bol classification; period-section difference is a derivation; frame and symbol determine the operator; equivariance priced at one class; Čech acyclicity reduced to Banach stages	Two multiplicative sections per chart, nonzero coordinate difference, one horizontal frame lift
B	Local first-order and rectangular kernels; Čech-cone, filtered, and concentrated reductions; three-check criterion	The classical-to-analytic chain map, chart comparison, boundary acyclicity
C	Local Cartier contraction (integral and mod-p); homotopy and mixed nilpotence; module-identity form; slope reduction of regular-weight acyclicity; support consequence derived	Separable module identity, global graded vanishing, conductor and descent compatibility, the slope input
D	*All* patched-level statements: existence of patched complex, action, augmentation; depth; freeness; descent isomorphism	Singular finite-level coherent complexes, action, augmentation, defect-one count, nonvanishing, weight-flatness
E	Seven rigidity routes; exact two-clause characterization; finite Frobenius tests; finiteness of the ramification locus	One arithmetic transversality input at x
F	Componentwise faithfulness; depth-support; MCM clause deleted; dense regular-weight support route; reduced derived criterion	One surviving automorphic witness on every integral component (or dense regular-weight support), plus control

The implication graph is acyclic:

$$\boxed{\mathrm{C}+\mathrm{D}} \Longrightarrow \boxed{R_x = A_{\mathrm{full},x} = A_{\mathrm{old},x} \text{ finite flat over } \Lambda_x},$$

$$\boxed{\mathrm{E}} \Longrightarrow \boxed{\Lambda_x = R_x = A_{\mathrm{full},x} = A_{\mathrm{old},x}},$$

and only afterward

$$\boxed{\mathrm{A}+\mathrm{B}} \Longrightarrow \boxed{\text{local weight-}(1,k)\text{ classicality at } x}.$$

The integral all-components conclusion additionally requires F.

11. Proof of the conditional synthesis

Theorem 11.1 (Conditional local one-ring and classicality theorem)**.** *The conclusions of Theorem 1.2 hold under the conjectures stated there.*

Proof. Conjecture C first proves, before completion, that the ordinary old and full cohomologies agree and that x lies on the old support, by Proposition 10.1; so Warning 2.3 no longer applies, and $A_{\mathrm{full},x} \cong A_{\mathrm{old},x}$.

Conjecture D supplies the finite-level data of Criterion 7.15. Theorem 7.9 with Proposition 7.10 produces the patched complex, action, and augmentation; Corollary 7.11 gives patched freeness and the identification $R^{c}/\mathfrak{a}R^{c} \cong T_0$; and clauses (D3)–(D4) of the conjecture descend these along Proposition 7.14 to

$$R_x \cong A_{\mathrm{old},x},$$

with $A_{\mathrm{old},x}$ finite flat over Λ_x. Conjecture E, through Theorem 8.1 and any route of Proposition 8.17, gives $A_{\mathrm{old},x} \cong \Lambda_x$. The local one-ring conclusion follows by Theorem 8.22.

Assume now Conjectures A and B. The one-ring conclusion identifies the full and old local branches. On the old branch, Conjecture A identifies the partial operators with $\partial_{z_{\tau_1}}$ and $\partial_{z_{\tau_2}}^{k-1}$ up to nonzero scalars. Theorem 4.2 and Proposition 4.3 compute the local joint kernel. Conjecture B identifies its global complex with the classical coherent complex, giving

$$H^{(1,k)}_{\mathrm{cl}} = \ker(\mathcal{N}_{\tau_1}) \cap \ker(D_{\tau_2,k}).$$

Finally assume Conjecture F. Its clause (I1) makes the patched action faithful by Theorem 9.1, so the integral deformation-to-Hecke surjection is an isomorphism. (The maximal Cohen–Macaulayness that earlier formulations of the conjecture assumed is supplied, where wanted for the one-point-per-component criterion of Theorem 9.2, by Theorem 9.3 from Conjecture D's outputs.) Its fixed-weight control and local–global compatibility clauses identify every point satisfying the deformation conditions with a classical weight-$(1,k)$ automorphic eigensystem. This is the asserted integral modularity consequence. □

Remark 11.2 (How conditionality should be presented). A conditional paper should not write the desired global theorem as if proved and then add a late disclaimer. The professional form is the one used here: state the open inputs individually and minimally, prove an explicit conditional synthesis, separate all unconditional results, and explain both the evidence and the obstructions. A future proof of any input can then be inserted without changing the logic of the paper.

12. Avenues for further research

The six conjectures are not equally remote. Each has a sharply localized next target, a proved reduction that shrinks it, and independent value on its own; we record for each why we hold it and why we consider it a fruitful frontier.

12.1. The period-extension construction of the singular operator. The most direct route to Conjecture A is to construct, on each Hodge–Tate chart, the étale and de Rham multiplicative sections of the square-zero period extension. By Lemma 3.8, their difference is already the desired derivation, and by Lemma 3.6 it remains only to calculate

$$\sigma_{\mathrm{et}}(z_i) - \sigma_{\mathrm{dR}}(z_i) = c_i \neq 0$$

and to construct one horizontal lift of the automorphic frame. We hold the conjecture because its exact one-variable analogue is a theorem of Pan [16, 17], because the general Sen-theoretic machinery exists [18, 14], and because the Bol classification (Theorem 3.2) shows the conjecture's infinite-level shape is forced rather than accidental: the classical avatar exists, is second order, and is unique, so a first-order operator has exactly one place to live. The avenue is fruitful because it converts an abstract global operator into two local period calculations and one Čech descent problem, and success would produce a singular partial Fontaine operator of independent interest, before and independently of any patching. Proposition 3.4 also directs the search: looking for an algebraic square root of $\mathcal{B}_1$ cannot succeed, so the construction must genuinely use the period sheaf.

12.2. **Filtered and rectangular comparison.** The likely economical route to Conjecture B is to filter the singular analytic and coherent complexes so that the associated graded maps are regular-weight comparisons; Proposition 5.9 then upgrades graded quasi-isomorphisms to a quasi-isomorphism of the total complexes, and much of the regular-weight technology is available [13, 20]. Alternatively, under localization to a single degree, Proposition 5.8 reduces the whole conjecture to one module isomorphism. The descended, second-order form of the comparison is already a theorem given three concrete checks (Theorem 4.11); the rectangle of Theorem 4.4, with its edge–Casimir structure (Proposition 4.6), is the local model and explains the expected form-plus-companion doubling at the singular place. We hold the conjecture because its modular-curve instance is Pan's classicality theorem and its regular-parallel Hilbert instance is Jiang's; what makes it a fruitful frontier is that the singular filtration and boundary compatibility are the only genuinely new ingredients.

12.3. **A module-level branch identity.** For Conjecture C, the strongest reduction is obtained when the localized old complexes have cohomology in one degree: then every homotopy obligation is a module identity (Proposition 5.8), and the geometric target becomes

$$U_{\mathrm{sep}} = 0 \quad \text{on } H^d(C_{\mathrm{old}}),$$

together with mod-p graded vanishing for the Frobenius and conductor pieces — exactly what the Cartier engine computes chart-locally — and the classical positive-new-slope input for (C5). We hold the conjecture because its local half is proved here in two independent forms (Theorem 6.1, Proposition 6.5), because its étale shadow is the Eichler–Shimura congruence relation, and because Diamond's saving trace [8] proves precisely the Iwahori-level degeneracy-map geometry the intertwining homotopies require, over the stratification of [20, 9]. The avenue is fruitful because a proof would be a new mod-p theorem of independent interest — a derived Goren–Oort branch-elimination theorem — and, combined with Conjecture D, it yields the one-ring theorem. It is the recommended first target.

12.4. **Finite-level coherent patching at the singular fibre.** The patched-level commutative algebra is complete (Section 7); the task is emphatically *not* to rebuild patching, but to place the higher-Hida complexes [11, 10] at the singular specialization in a weak amplitude-$[0, 1]$ datum with an action on top cohomology and an augmentation quasi-isomorphism — the natural outputs of a cohomology theory, as Proposition 7.10 was designed to accept — and to verify the defect-one count, nonvanishing, and weight-flatness. We hold the conjecture because the elliptic weight-one analogue of the entire mechanism is carried out by Calegari–Geraghty [5], because the characteristic-zero shadow of the defect-one count is proved here (Proposition 8.4), and because nonvanishing is a single ordinary class. This is the closest item to completion: it has a concrete finite-level checklist, and by itself it would produce a coherent defect-one $R = \mathbb{T}$ theorem on the old branch.

12.5. **Arithmetic transversality at one point.** Conjecture E has seven attack routes. The most computational is Corollary 8.10: finitely many Frobenius traces separate the tangent space, so the conjecture is decided by finitely many derivatives. The most spectral is route (R4): a monogenic Hecke presentation with simple residual root. The most arithmetic is route (R5): one ordinary family through x with unit adjoint congruence ideal, potentially through an adjoint p-adic L-value in the spirit of Hida's congruence-module theory [12]. The most automorphic are the fibre route (R7): reducedness of the fixed-weight Hecke fibre is a multiplicity-one statement approachable by q-expansion methods. We hold the conjecture at a suitable point because the eigencurve analogue is a theorem [2] and failure is confined, after generic étaleness, to finitely many points (Proposition 8.19); we present it honestly with its counter-evidence, since RM points do ramify [3] and Counterexample 8.20 shows no formal proof exists. Proposition 8.18 makes the frontier exactly two finite checks: one dimension bound and one nonzero derivative.

12.6. **One automorphic witness on each integral component.** For Conjecture F, Theorem 9.2 reduces an entire component to one point away from component intersections, so the missing input is a level-raising, Ihara-type, or potential-automorphy argument constructing one nonzero classical specialization on every minimal component with the prescribed bad-place conditions. Proposition 10.3 offers a complementary route located squarely in known territory: finite flatness plus automorphy above a *dense set of regular weights* — where ordinary modularity-lifting theorems already operate — forces support at every minimal prime; and for reduced, field-valued conclusions, Theorem 7.16 lowers the bar further, to one nonzero derived fibre per minimal prime, with no depth or freeness input at all. We hold the conjecture because component-support control is achieved in proved integral $R = \mathbb{T}$ theorems at regular weight, and because Counterexample 9.6 proves the hypothesis is sharp rather than an artifact. The avenue is fruitful because it is finite: once one witness per component is found, no further commutative algebra is needed.

12.7. **Recommended order.** The most independent targets are Conjecture A and the module identity in Conjecture C. The former could anchor a singular-operator and classicality paper; the latter a derived Goren–Oort branch-elimination paper. The finite-level datum of Conjecture D is the direct route to a coherent $R = \mathbb{T}$ theorem, and its checklist is the shortest. Local transversality should be pursued in parallel, because any one of the seven rigidity routes closes the one-ring argument once branch separation and patching are available; and the dense-regular-weight route to integral support should be developed alongside, since it substitutes known regular-weight modularity for new singular-weight constructions wherever a reduced conclusion suffices.

Appendix A. Finite-length verifications

Four scripts accompany the paper. None is used in any proof. All random sampling is seeded deterministically inside the scripts, so every reported number is exactly reproducible; the negative controls are diagnostically informative, showing that the load-bearing hypotheses are not decorative. For each script we state exactly which clauses are tested and in what generality; no script tests a theorem in its full stated generality, and none is claimed to.

Availability and integrity. The scripts are distributed as ancillary files with the arXiv posting of this article. They require only Python 3.8 or later and its standard library; each is run as `python3 <script>` with no arguments and prints the figures quoted below. Independently of where a copy is obtained, the following SHA-256 hashes pin the exact versions used for the runs reported here:

```
cartier_check.py
  96a98291e939b01f6896583168dd99a15f6f4dc01066547f4487fb3bb187bb2b
cartier_congruence_check.py
  4052f52cb613a162d11263553e61e3c65034e80739a6e750d3434fe845dede98
edge_casimir_check.py
  e7040db55d47bf03dece2d0131b318f86326f0867820fa8920798a42099dc748
mixed_contraction_check.py
  758d3e55552951b894a750928d71bced89237b951d7ad8494851b17ec0c9f5b9
```

A.1. **Cartier congruences.** The script `cartier_check.py` tests only the trace inclusions (b) and the unit-coefficient congruence behind (e) of Theorem 6.1 — not parts (c)–(f), not the matrix theorem, and not tangential variables — after base change to

$$A = \mathbf{Z}/p^e\mathbf{Z}, \qquad S = A[t]/(t^{pK}), \qquad R = A[x]/(x^K), \qquad x \mapsto u_0 t^p.$$

This base change is exact for precisely these assertions: the inclusions tested are statements about traces of multiplication matrices in the basis $1, t, \ldots, t^{p-1}$, and both the basis property and the matrices are compatible with the quotients by (p^e) and (t^{pK}), so each tested congruence in the truncated model is the literal image of the corresponding congruence in the theorem. The script decomposes S in the basis, constructs the multiplication matrices, and computes their traces exactly in the finite rings, with a random unit u_0 subject to (6.1) (seeds fixed per prime). With $K = 4$, $e = 3$, and forty trials for each prime, a fresh run reproduced:

Test	(b1)	(b2)	(e)
$p = 3$, hypothesis satisfied	40/40	40/40	40/40
$p = 5$, hypothesis satisfied	40/40	40/40	40/40
$p = 3$, hypothesis violated	2/40	0/40	2/40

Here (b1) denotes $\mathrm{Tr}(S) \subset pR$, (b2) denotes $\mathrm{Tr}(tS) \subset pxR$, and (e) denotes $\mathrm{Tr}(g) \equiv pg(0) \pmod{pxR}$.

A.2. **The congruence proposition and its Laurent counterexample.** The script `cartier_congruence_check.py` is a deterministic regression test for Proposition 6.5, run with exact integer arithmetic on finite Laurent polynomials (no truncation and no sampling), with the trace computed honestly from the $p \times p$ multiplication matrix rather than from the closed formula. For $p = 5$, $K = 4$, and fixed integral data $w_0 \in \mathbf{Z}[x]$, $W_1 \in \mathbf{Z}[t]$, it verifies the integrality of U_W on $\mathbf{Z}[x]$, the p-divisibility of $U_W(1)$, and the resonance congruence modulo p for all poles $m \le 3p$; and it then runs the negative control $W_1 = t^{1-p}$ of Remark 6.6, confirming $U_W(1) = p\,x^{-1}$, outside the integral lattice. A fresh run passed the repaired clauses and reproduced the counterexample exactly.

A.3. **Edge–Casimir identity.** The script `edge_casimir_check.py` verifies the operator identity $F_\lambda E = -T(T-1-\lambda)$ of Proposition 4.6 on polynomials of degree up to 14 for integer parameters $-4 \le \lambda \le 4$, with exact rational arithmetic; this is the entire content of the displayed identity on polynomial test functions. A fresh run passed all checks.

A.4. **Mixed contraction.** The script `mixed_contraction_check.py` builds a seeded random 6×6 matrix over $\mathbf{Z}/5^8$ that is strictly upper triangular modulo 5 (filtration-lowering modulo p with arbitrary p-divisible errors, the module hypothesis of Theorem 5.11 with $a + 1 = 6$) and verifies $U^{6n} \equiv 0 \pmod{5^n}$ for $1 \le n \le 5$; the homotopy form of Theorem 5.12 is not exercised. A fresh run confirmed each divisibility at the predicted exponent.

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