TATE FRACTURE, ORIENTATION OBSTRUCTIONS, AND GENUINE C_2-COEFFICIENTS IN BERKOVICH MOTIVES

RIYAN CHRISTY

Abstract. Let $\Gamma = C_2$. For an arbitrary presentable stable ∞-category $\mathcal{C}$, we prove a coefficientwise Tate fracture theorem: genuine Γ-objects with coefficients in $\mathcal{C}$ are equivalent to triples

$$(B, G, \gamma), \qquad B \in \mathrm{Fun}(B\Gamma, \mathcal{C}), \quad G \in \mathcal{C}, \quad \gamma : G \longrightarrow B^{t\Gamma}.$$

When $\mathcal{C}$ is presentably symmetric monoidal, this equivalence is symmetric monoidal. Writing $R_{\mathcal{C}} = (\mathbf{1}_{\mathcal{C}})^{t\Gamma}$, we deduce that a line $L \in \mathrm{Pic}(\mathcal{C})$ admits the correction needed to replace an $L[-1]$-valued geometric shadow by $\mathbf{1}[-1]$ precisely when

$$R_{\mathcal{C}} \simeq L \otimes R_{\mathcal{C}}$$

as $R_{\mathcal{C}}$-modules. The space of choices is the corresponding equivalence space and, when nonempty, is a torsor under $\mathrm{GL}_1(R_{\mathcal{C}})$. We call such a trivialization a Tate orientation.

We apply this criterion to the sphere-valued Berkovich motives recently introduced by Scholze, whose cancellation theorem holds over every base. For every small arc-stack X, the canonical effective genuine object

$$\mathbb{T}_X^{\mathrm{cw}} = \mathbb{T}_X \boxtimes S^{\sigma-1}$$

has Borel shadow $\mathbb{T}_X$ with additive sign action, geometric-fixed-point shadow $\mathbb{T}_X[-1]$, invertible stable image, and satisfies integral effective cancellation. A normalized stable line with the same Borel shadow and geometric shadow $\mathbf{1}_X[-1]$ exists if and only if $\mathbb{T}_X$ is Tate-orientable. We construct a real Betti–Galois realization under which $\mathbb{T}_{\mathbb{R}}$ becomes the sign local system. Lin's form of the Segal conjecture then gives a 2-adic monodromy obstruction, proving that no normalized stable or effective line exists over the real point, or over any base admitting a real point. In contrast, normalized effective objects exist at the complex point, after inverting 2, and at nonarchimedean geometric points of residue characteristic 2. Finally, we state an exact effective-lifting criterion, separating the unconditional stable classification from the remaining essential-image problem.

Contents

Date: July 14, 2026.
2020 *Mathematics Subject Classification.* Primary 55P91, 14F42; Secondary 18N70.
Key words and phrases. Berkovich motives, genuine equivariant spectra, Tate construction, cancellation, Picard obstruction, real realization.

1. Introduction

Let X be a small arc-stack in the sense of Scholze. We write

$$\mathcal{C}_X^{\mathrm{eff}} = \mathcal{D}_{\mathrm{mot}}^{\mathrm{eff}}(X; \mathrm{Sp})$$

for the sphere-valued effective Berkovich motivic category, $\mathbb{T}_X \in \mathcal{C}_X^{\mathrm{eff}}$ for its first Tate object, and

$$\mathcal{C}_X = \mathcal{D}_{\mathrm{mot}}(X; \mathrm{Sp}) = \mathcal{C}_X^{\mathrm{eff}}[\mathbb{T}_X^{-1}]$$

for its stabilization. Scholze's construction is written primarily with integral Eilenberg–Mac Lane coefficients, but the coefficient spectrum may be replaced by the sphere spectrum, or more generally by an arbitrary E_∞-ring. His cancellation theorem states that

$$- \otimes \mathbb{T}_X : \mathcal{C}_X^{\mathrm{eff}} \longrightarrow \mathcal{C}_X^{\mathrm{eff}} \tag{1}$$

is fully faithful for every small arc-stack [14, Def. 5.2, footnote 3, Thm. 7.1]. These facts are inputs to the present paper.

Two phenomena motivate adjoining genuine C_2-coefficients. The first is hermitian: real algebraic K-theory packages algebraic K-theory with its duality action, Grothendieck–Witt theory, and L-theory into a single genuine C_2-spectrum, whose Borel, genuine-fixed-point, and geometric-fixed-point pieces recover the three theories [4, 5]; a motivic theory with genuine C_2-coefficients is the natural receptacle for such objects. The second is the contrast with $\mathbb{A}^1$-homotopy theory, where quadratic-form phenomena enter the sphere itself and inversion on $\mathbb{G}_m$ acts on the Tate object by $-\langle -1\rangle$ rather than by -1 [11]. Berkovich motives behave instead like étale or topological theories—the first Tate object over the complex point is the unit—so one expects their genuine C_2-refinement to behave like equivariant topology. The results below make precise exactly how much of that expectation is formal, and what obstructs the remainder.

We fix a coefficient group

$$\Gamma = C_2$$

and study

$$\mathcal{C}_X^{\mathrm{eff},g} = \mathcal{C}_X^{\mathrm{eff}} \otimes \mathrm{Sp}^{g\Gamma}, \qquad \mathcal{C}_X^{g} = \mathcal{C}_X \otimes \mathrm{Sp}^{g\Gamma}.$$

The coefficient group Γ is independent of any group arising from geometric descent. In particular, at the real point we also use $G_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and the two copies of C_2 play different roles.

The structural input is a coefficientwise Tate-fracture theorem. For a presentable stable category $\mathcal{C}$, let

$$\mathrm{Frac}_\Gamma(\mathcal{C}) = \left\{ (B, G, \gamma) \;\middle|\; B \in \mathrm{Fun}(B\Gamma, \mathcal{C}),\ G \in \mathcal{C},\ \gamma : G \to B^{t\Gamma} \right\}.$$

This is the pullback

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \times_{\mathrm{ev}_1, \mathcal{C}, (-)^{t\Gamma}} \mathrm{Fun}(B\Gamma, \mathcal{C}).$$

Theorem A (Theorem 2.6)**.** *For every presentable stable ∞-category $\mathcal{C}$, there is a natural equivalence*

$$\mathcal{C} \otimes \mathrm{Sp}^{g\Gamma} \simeq \mathrm{Frac}_\Gamma(\mathcal{C}).$$

For the genuine object corresponding to (B, G, γ), the Borel shadow is B, the geometric-fixed-point shadow is G, and the genuine fixed points are

$$B^{h\Gamma} \times_{B^{t\Gamma}} G.$$

If $\mathcal{C}$ is presentably symmetric monoidal and tensor product preserves colimits separately, the equivalence is symmetric monoidal.

The proof is direct. In the spectral Mackey model, genuine fixed points are an evaluation and therefore preserve colimits; consequently the entire isotropy recollement of genuine C_2-spectra—the free–geometric idempotent triangle, the Borel and geometric identifications, the Adams isomorphism, and the norm—transports along $\mathcal{C} \otimes (-)$, every ingredient being colimit-preserving. Reconstruction is then an elementary semiorthogonal gluing argument, and the Tate construction enters through an internal mapping-space computation. This avoids any unsupported assertion that tensoring with an arbitrary presentable category preserves a right-lax limit, and it gives the symmetric monoidal formula transparently from the lax symmetric monoidality of the Tate construction. The theorem is a coefficientwise C_2-form of classical isotropy separation [13, §II.4]. Unbounded and multiplicative refinements at $\mathcal{C} = \mathrm{Sp}$ appear in [6, 12], and closely related reconstructions for rigidly compactly generated coefficient categories in [1, 2]; we are not aware of a reference at the present generality, and the gluing map is made explicit for later use.

The fracture theorem converts the normalization problem into a Picard-theoretic obstruction. Put

$$R_{\mathcal{C}} := (\mathbf{1}_{\mathcal{C}})^{t\Gamma} \in \mathrm{CAlg}(\mathcal{C}).$$

Theorem B (Theorem 3.1)**.** *Let $L \in \mathrm{Pic}(\mathcal{C})$. The space of framed genuine correction lines with Borel shadow $\mathbf{1}$ and geometric shadow L^{-1} is naturally equivalent to*

$$\mathrm{Eq}_{\mathrm{Mod}_{R_{\mathcal{C}}}(\mathcal{C})}\big(R_{\mathcal{C}}, L \otimes R_{\mathcal{C}}\big).$$

Consequently such a line exists if and only if $L \otimes R_{\mathcal{C}}$ is the trivial invertible $R_{\mathcal{C}}$-module. When the space is nonempty, it is a torsor under $\mathrm{GL}_1(R_{\mathcal{C}})$. Orientations are functorial under symmetric monoidal colimit-preserving functors.

We call an equivalence $R_{\mathcal{C}} \simeq L \otimes R_{\mathcal{C}}$ a *Tate orientation* of L. This formulation separates three logically different assertions: existence of a stable line, choice of its coherent gluing datum, and existence of an effective lift. Equality in a Grothendieck group, or even equality of the two shadows, does not supply the gluing datum.

The Berkovich application begins with a canonical object. Let σ be the sign representation of Γ, and let $S^{\sigma-1}$ be the corresponding virtual representation sphere.

Theorem C (Theorem 4.4)**.** *For every small arc-stack X, the effective object*

$$\mathbb{T}_X^{\mathrm{cw}} = \mathbb{T}_X \boxtimes S^{\sigma-1} \in \mathcal{C}_X^{\mathrm{eff},g}$$

has Borel shadow $\mathbb{T}_X$ with additive sign action and geometric-fixed-point shadow $\mathbb{T}_X[-1]$. Its stable image is invertible, and tensoring with $\mathbb{T}_X^{\mathrm{cw}}$ is fully faithful on $\mathcal{C}_X^{\mathrm{eff},g}$, integrally.

The word *normalized* refers to replacing the geometric shadow $\mathbb{T}_X[-1]$ by $\mathbf{1}_X[-1]$ while retaining the Borel shadow. By Theorem B, a normalized stable line exists precisely when

$$R_X \simeq \mathbb{T}_X \otimes R_X, \qquad R_X = (\mathbf{1}_X)^{t\Gamma}.$$

This criterion is not automatically satisfied.

Theorem D (Theorems 5.1, 6.3, and 6.5)**.** *The space of framed normalized stable genuine Tate lines over X is*

$$\mathrm{Eq}_{\mathrm{Mod}_{R_X}(\mathcal{C}_X)}\big(R_X, \mathbb{T}_X \otimes R_X\big).$$

Over the real archimedean point, this space is empty. More generally, it is empty over every base admitting a real point. Hence no normalized stable or effective genuine Tate line exists on such a base.

The obstruction is detected after real Betti–Galois realization. The first Tate object becomes the sphere with sign $G_{\mathbb{R}}$-monodromy, whereas the Tate algebra becomes a constant local system with fiber $\mathbb{S}^{t\Gamma} \simeq \mathbb{S}_2^\wedge$ by Lin's form of the Segal conjecture. An equivalence between the constant free rank-one module and its sign twist would give a unit $a \in \mathbb{Z}_2^\times$ satisfying $a = -a$, which is impossible.

There are also unconditional positive cases. At the complex point, $\mathbb{T}_X \simeq \mathbf{1}_X$, so the canonical coefficientwise object is already normalized. After inverting 2, the Tate construction vanishes and the fracture category splits; this produces a normalized effective object whose stable image is invertible. The same conclusion holds at nonarchimedean geometric points of residue characteristic 2, because 2 is invertible in the corresponding motivic category.

Logical status. All principal theorems in this paper are unconditional. The external inputs are published theorems used with precise citations: Scholze's cancellation theorem and archimedean comparison [14], Lin's theorem [8], and the classical isotropy package for genuine C_2-spectra [7, 10, 13]. For a general base, we do *not* assert that a normalized line exists. Stable existence is classified exactly by the Tate-orientation space. Effective existence, when a stable orientation exists, is a separate essential-image question stated precisely in Proposition 7.2. Thus no hypothesis is hidden in a proof or relegated to an informal caveat: conditional conclusions are stated as implications with their hypotheses in the theorem statement, and the unresolved general case is recorded as Problem 7.8.

Organization. Section 2 proves the coefficientwise fracture theorem and its symmetric monoidal refinement. Section 3 develops Tate orientations, base change, and the monodromy obstruction. Section 4 constructs the canonical coefficientwise Berkovich Tate object and proves effective cancellation. Section 5 applies the orientation criterion. Section 6 constructs the real realization and proves the real obstruction. Section 7 treats effective lifts and the positive regimes.

2. Coefficientwise C_2-Tate fracture

Throughout this section, $\mathcal{C}$ is a presentable stable ∞-category and $\Gamma = C_2$. For $B \in \mathrm{Fun}(B\Gamma, \mathcal{C})$, write

$$N_B : B_{h\Gamma} \longrightarrow B^{h\Gamma}$$

for the norm and

$$B^{t\Gamma} = \mathrm{cofib}(N_B)$$

for the Tate construction. Here $(-)_{h\Gamma}$ and $(-)^{h\Gamma}$ are the colimit and limit over $B\Gamma$ formed internally to $\mathcal{C}$: the constant-diagram functor $\mathrm{triv} : \mathcal{C} \to$

$\mathrm{Fun}(B\Gamma, \mathcal{C})$ has left adjoint $(-)_{h\Gamma}$ and right adjoint $(-)^{h\Gamma}$. We normalize the norm as the mate, under $\mathrm{triv} \dashv (-)^{h\Gamma}$, of the canonical transformation $\mathrm{triv} \circ (-)_{h\Gamma} \to \mathrm{id}$ of endofunctors of $\mathrm{Fun}(B\Gamma, \mathcal{C})$ obtained from summation over the group; for $\mathcal{C} = \mathrm{Sp}$ this recovers the classical norm [13, §I.1]. The advantage of this normalization is that its generating datum is a transformation between colimit-preserving functors, so it transports along $\mathcal{C} \otimes (-)$.

2.1. Mackey objects and norm factorizations.

Let $A^{\mathrm{eff}}(\Gamma)$ denote the effective Burnside ∞-category. Define

$$\mathrm{Mack}_\Gamma(\mathcal{C}) = \mathrm{Fun}^{\oplus}\big(A^{\mathrm{eff}}(\Gamma)^{\mathrm{op}}, \mathcal{C}\big).$$

The coefficientwise spectral Mackey theorem gives a natural equivalence

$$\mathcal{C} \otimes \mathrm{Sp}^{g\Gamma} \simeq \mathrm{Mack}_\Gamma(\mathcal{C}). \tag{2}$$

This is obtained from the spectral Mackey model $\mathrm{Sp}^{g\Gamma} \simeq \mathrm{Mack}_\Gamma(\mathrm{Sp})$ of [3], together with the functor-category formula for tensoring with the compactly generated category $\mathrm{Sp}^{g\Gamma}$; in the form needed here, for an arbitrary presentable stable $\mathcal{C}$, it is [1, Obs. 1.9]. In the Mackey model, evaluation at the free orbit is the Borel functor u, evaluation at the fixed orbit is the genuine fixed points $(-)^\Gamma$, and both preserve limits and colimits; geometric fixed points Φ^Γ are the cofiber of the transfer $(uX)_{h\Gamma} \to X^\Gamma$ and hence also preserve colimits.

Let $\mathrm{Fact}_N(\mathcal{C})$ be the ∞-category whose objects are diagrams

$$B_{h\Gamma} \xrightarrow{\alpha} E \xrightarrow{\beta} B^{h\Gamma} \tag{3}$$

with $B \in \mathrm{Fun}(B\Gamma, \mathcal{C})$ and $\beta\alpha = N_B$.

The relation between Mackey objects and norm factorizations will be established as Corollary 2.8 below, as a consequence of the fracture theorem; it is not an input to its proof.

2.2. The isotropy recollement.

Recall the idempotent triangle of pointed genuine Γ-spaces

$$(E\Gamma)_+ \longrightarrow S^0 \longrightarrow \widetilde{E}\Gamma,$$

smashed into $\mathrm{Sp}^{g\Gamma}$. Write $a = (E\Gamma)_+ \otimes (-)$ and $b = \widetilde{E}\Gamma \otimes (-)$, so that there is a functorial triangle $a \to \mathrm{id} \to b$ of exact endofunctors of $\mathrm{Sp}^{g\Gamma}$, with a, b idempotent and $ab \simeq 0 \simeq ba$. The organizing principle of what follows is that *every functor we transport preserves colimits*, so that no limit construction is ever tensored; the limits that occur are formed internally, after tensoring.

Lemma 2.1 (The spectrum-level package)**.** *The following hold in* $\mathrm{Sp}^{g\Gamma}$*, and every displayed functor preserves colimits.*

(i) *The essential image of* a *(the* free part*) is equivalent to* $\mathrm{Fun}(B\Gamma, \mathrm{Sp})$ *via the Borel functor* u*; write* $j_!$ *for the inverse equivalence followed by the inclusion, so that* $u\, j_! \simeq \mathrm{id}$.

(ii) *The essential image of* b *(the* geometric part*) is equivalent to* Sp *via* Φ^Γ*; write* $i_*(-) = \widetilde{E}\Gamma \otimes \mathrm{infl}(-)$ *for the inverse equivalence followed by the inclusion, so that* $\Phi^\Gamma i_* \simeq \mathrm{id}$ *and* $(-)^\Gamma \circ i_* \simeq \mathrm{id}$.

(iii) *$u\circ b \simeq 0$; in particular $u\, i_* \simeq 0$.*
(iv) *$a\circ \mathrm{infl} \simeq j_! \circ \mathrm{triv}$ and $b \circ \mathrm{infl} \simeq i_*$.*
(v) (Adams isomorphism) *There is a natural equivalence $((-)^{\Gamma}) \circ j_! \simeq (-)_{h\Gamma}$ of functors $\mathrm{Fun}(B\Gamma, \mathrm{Sp}) \to \mathrm{Sp}$, and under it the canonical comparison $(j_! Y)^{\Gamma} \to (u\, j_! Y)^{h\Gamma} \simeq Y^{h\Gamma}$ is identified with the norm N_Y.*

These are classical; see [7, Ch. II], *and* [10, §6], [13, §§I.1–I.2] *for modern accounts. Here* infl *denotes inflation, left adjoint to the genuine fixed points $(-)^{\Gamma}$; in the Mackey model both functors preserve colimits.*

Lemma 2.2 (The coefficientwise package)**.** *Let $\mathcal{C}$ be presentable stable. All statements of Lemma 2.1 hold verbatim in $\mathcal{C} \otimes \mathrm{Sp}^{g\Gamma}$, with $\mathrm{Fun}(B\Gamma, \mathcal{C})$ and $\mathcal{C}$ in place of $\mathrm{Fun}(B\Gamma, \mathrm{Sp})$ and Sp, the limits over $B\Gamma$ being formed internally to $\mathcal{C}$. Moreover, inflation remains left adjoint to genuine fixed points, and* triv *remains left adjoint to the internal $(-)^{h\Gamma}$.*

Proof. The 2-functor $\mathcal{C} \otimes (-)$ on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ preserves equivalences and functorial triangles of colimit-preserving functors, idempotency, and adjunctions both of whose members preserve colimits. Statements (i)–(iv) are of this kind, using $\mathcal{C} \otimes \mathrm{Fun}(B\Gamma, \mathrm{Sp}) \simeq \mathrm{Fun}(B\Gamma, \mathcal{C})$ and $\mathcal{C} \otimes \mathrm{Sp} \simeq \mathcal{C}$; so is the adjunction $\mathrm{infl} \dashv (-)^{\Gamma}$, since genuine fixed points preserve colimits in the Mackey model. The adjunction $\mathrm{triv} \dashv (-)^{h\Gamma}$ needs no transport: $\mathrm{Fun}(B\Gamma, \mathcal{C})$ is a functor category, in which the constant-diagram functor tautologically has the internal limit as right adjoint.

For (v), both sides of the asserted equivalence are colimit-preserving, so the equivalence transports. For the identification of the comparison map with the norm, pass to mates under $\mathrm{triv} \dashv (-)^{h\Gamma}$: the norm was normalized above as the mate of a transformation between colimit-preserving endofunctors, and the comparison map is likewise the mate of a transformation between colimit-preserving functors, obtained from the counit $a \to \mathrm{id}$ and (iv). The identity between the two mates is a spectrum-level fact and an identity between tensored transformations of colimit-preserving functors; it therefore descends to $\mathcal{C}$-coefficients. $\square$

2.3. **Reconstruction and Tate gluing.**

Recall the fracture category

$$\mathrm{Frac}_{\Gamma}(\mathcal{C}) = \left\{ (B, G, \gamma) \mid \gamma : G \to B^{t\Gamma} \right\}.$$

We first record two lemmas: an elementary reconstruction principle, and the computation in which the Tate construction appears.

Lemma 2.3 (Semiorthogonal reconstruction)**.** *Let $\mathcal{X}$ be a stable ∞-category with a functorial triangle $a \to \mathrm{id} \to b$ of exact endofunctors such that a, b are idempotent, $ab \simeq 0 \simeq ba$, and $\mathrm{Map}_{\mathcal{X}}(aX, bY) \simeq 0$ for all X, Y. Write $\mathcal{U}, \mathcal{Z} \subseteq \mathcal{X}$ for the essential images of a and b, and let $\mathcal{D}$ be the ∞-category of triples (U, Z, δ) with $U \in \mathcal{U}$, $Z \in \mathcal{Z}$, and $\delta : Z \to \Sigma U$. Then*

$$X \longmapsto \big(aX,\ bX,\ \delta_X : bX \to \Sigma aX\big),$$

with δ_X the connecting map of the triangle, is an equivalence $\mathcal{X} \simeq \mathcal{D}$, with inverse $(U, Z, \delta) \mapsto \operatorname{fib}(\delta)$.

Proof. For essential surjectivity, let (U, Z, δ) be a triple and $X = \operatorname{fib}(\delta)$, giving a fiber sequence $X \to Z \to \Sigma U$. Applying the exact functor a, and using $aZ \simeq 0$ (as $Z \in \operatorname{im} b$ and $ab \simeq 0$) together with $a\Sigma U \simeq \Sigma U$, gives $aX \simeq \Omega\Sigma U \simeq U$; applying b, and using $b\Sigma U \simeq 0$, gives $bX \simeq bZ \simeq Z$. The octahedron axiom applied to $X \to Z \to \Sigma U$ identifies the connecting map of $aX \to X \to bX$ with δ; conversely $\operatorname{fib}(\delta_X) \simeq X$ by rotating the defining triangle.

For full faithfulness, fix X, Y. Applying $\operatorname{Map}(-, Y)$ to $aX \to X \to bX$ gives a fiber sequence $\operatorname{Map}(bX, Y) \to \operatorname{Map}(X, Y) \to \operatorname{Map}(aX, Y)$; applying $\operatorname{Map}(aX, -)$ and $\operatorname{Map}(bX, -)$ to $aY \to Y \to bY$ gives $\operatorname{Map}(aX, Y) \simeq \operatorname{Map}(aX, aY)$ by the vanishing hypothesis, and a fiber sequence $\operatorname{Map}(bX, aY) \to \operatorname{Map}(bX, Y) \to \operatorname{Map}(bX, bY)$ with boundary landing in $\operatorname{Map}(bX, \Sigma aY)$. Assembling, $\operatorname{Map}(X, Y)$ is the pullback

$$\operatorname{Map}(aX, aY) \times_{\operatorname{Map}(bX, \Sigma aY)} \operatorname{Map}(bX, bY),$$

the two maps being postcomposition with δ_Y after Σa, and precomposition with δ_X. This is the mapping space in $\mathcal{D}$ between the associated triples. $\square$

Lemma 2.4 (The gluing mapping spaces)**.** *For $V \in \operatorname{Fun}(B\Gamma, \mathcal{C})$ and $z \in \mathcal{C}$, there is an equivalence, natural in both variables,*

$$\operatorname{Map}_{\mathcal{C} \otimes \mathrm{Sp}^{g\Gamma}}\big(i_* z,\ \Sigma j_! V\big) \simeq \operatorname{Map}_{\mathcal{C}}\big(z,\ V^{t\Gamma}\big).$$

Proof. Write $W = \Sigma j_! V$. By Lemma 2.2(iv), applying $\operatorname{Map}(-, W)$ to the triangle $j_! \operatorname{triv} z \to \operatorname{infl} z \to i_* z$ gives a fiber sequence

$$\operatorname{Map}(i_* z, W) \longrightarrow \operatorname{Map}(\operatorname{infl} z, W) \longrightarrow \operatorname{Map}(j_! \operatorname{triv} z, W).$$

The middle term is $\operatorname{Map}_{\mathcal{C}}(z, W^{\Gamma})$ by the inflation adjunction. For the right-hand term, $\operatorname{Map}(j_! Y, W) \simeq \operatorname{Map}(j_! Y, aW)$, because $\operatorname{Map}(j_! Y, bW) \simeq \operatorname{Map}_{\operatorname{Fun}(B\Gamma, \mathcal{C})}(Y, u\, bW) \simeq 0$ by Lemma 2.2(iii); since $u\, j_! \simeq \mathrm{id}$ and $u\, aW \simeq uW$, again by (iii) applied to the triangle $aW \to W \to bW$, this gives $\operatorname{Map}(j_! \operatorname{triv} z, W) \simeq \operatorname{Map}_{\operatorname{Fun}(B\Gamma, \mathcal{C})}(\operatorname{triv} z, uW) \simeq \operatorname{Map}_{\mathcal{C}}\big(z, (uW)^{h\Gamma}\big)$. Now specialize: $uW \simeq \Sigma V$, and $W^{\Gamma} \simeq \Sigma V_{h\Gamma}$ by the Adams isomorphism of Lemma 2.2(v), under which the comparison map $W^{\Gamma} \to (uW)^{h\Gamma}$ becomes ΣN_V. Hence

$$\operatorname{Map}(i_* z, \Sigma j_! V) \simeq \operatorname{Map}_{\mathcal{C}}\big(z, \operatorname{fib}(\Sigma N_V)\big) \simeq \operatorname{Map}_{\mathcal{C}}\big(z, V^{t\Gamma}\big),$$

using $\Sigma \operatorname{fib}(N_V) \simeq \operatorname{cofib}(N_V) = V^{t\Gamma}$. $\square$

Lemma 2.5 (Factorization versus fracture)**.** *There is a natural equivalence*

$$\operatorname{Fact}_N(\mathcal{C}) \simeq \operatorname{Frac}_{\Gamma}(\mathcal{C}).$$

It sends a norm factorization (3) to

$$G = \operatorname{cofib}(\alpha), \qquad \gamma : G \longrightarrow B^{t\Gamma}$$

induced by the commutative square

$$\begin{tikzcd} B_{h\Gamma} \arrow[r, "\alpha"] \arrow[d, "N_B"'] & E \arrow[d, "\beta"] \\ B^{h\Gamma} \arrow[r, equal] & B^{h\Gamma}. \end{tikzcd}$$

Its inverse sends (B, G, γ) *to*

$$E = B^{h\Gamma} \times_{B^{t\Gamma}} G.$$

Proof. The map of cofiber sequences associated with the displayed square gives $\gamma : \mathrm{cofib}(\alpha) \to \mathrm{cofib}(N_B) = B^{t\Gamma}$. Conversely, given $\gamma : G \to B^{t\Gamma}$, form the pullback

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\beta"'] & G \arrow[d, "\gamma"] \\ B^{h\Gamma} \arrow[r] & B^{t\Gamma}. \end{tikzcd}$$

The norm triangle makes the fiber of $B^{h\Gamma} \to B^{t\Gamma}$ equal to $B_{h\Gamma}$. Pullback therefore gives a fiber sequence

$$B_{h\Gamma} \xrightarrow{\alpha} E \longrightarrow G,$$

and the composite $B_{h\Gamma} \to E \to B^{h\Gamma}$ is N_B. Thus $\beta\alpha = N_B$ and $\mathrm{cofib}(\alpha) \simeq G$. The two constructions are inverse by the universal properties of fibers and cofibers in a stable category. □

Theorem 2.6 (Coefficientwise C_2-fracture)**.** *For every presentable stable* ∞*-category* $\mathcal{C}$*, there is a natural equivalence*

$$\Theta_{\mathcal{C}} : \mathcal{C} \otimes \mathrm{Sp}^{g\Gamma} \xrightarrow{\simeq} \mathrm{Frac}_\Gamma(\mathcal{C}). \tag{4}$$

If X *corresponds to* (B, G, γ)*, then*

$$uX \simeq B, \qquad \Phi^\Gamma X \simeq G, \qquad X^\Gamma \simeq B^{h\Gamma} \times_{B^{t\Gamma}} G.$$

Proof. The hypotheses of Lemma 2.3 hold in $\mathcal{C} \otimes \mathrm{Sp}^{g\Gamma}$ with the tensored idempotent triangle of Lemma 2.2: the required vanishing follows from $\mathrm{Map}(j_!Y, i_*z) \simeq \mathrm{Map}_{\mathrm{Fun}(B\Gamma,\mathcal{C})}(Y, u\, i_*z) \simeq 0$ by Lemma 2.2(iii). Thus $\mathcal{C} \otimes \mathrm{Sp}^{g\Gamma}$ is equivalent to the category of triples $(U, Z, \delta : Z \to \Sigma U)$ supported on the free and geometric parts. Identifying the free part with $\mathrm{Fun}(B\Gamma, \mathcal{C})$ via u, the geometric part with $\mathcal{C}$ via Φ^Γ, and converting the gluing data by the natural equivalence of Lemma 2.4, transforms this triple category into $\mathrm{Frac}_\Gamma(\mathcal{C})$.

For the shadows: $u\,b \simeq 0$ gives $uX \simeq u\,aX \simeq B$, and $\Phi^\Gamma a \simeq 0$ gives $\Phi^\Gamma X \simeq \Phi^\Gamma bX \simeq G$. For the fixed points, apply $(-)^\Gamma$ to the fiber sequence $X \to i_*G' \to \Sigma j_!B'$ produced by Lemma 2.3: by parts (ii) and (v) of Lemma 2.2,

$$X^\Gamma \simeq \mathrm{fib}\big(G \to \Sigma B_{h\Gamma}\big),$$

the map being the composite of $\gamma : G \to B^{t\Gamma}$ with the boundary $B^{t\Gamma} \to \Sigma B_{h\Gamma}$ of the norm triangle. Since $B^{h\Gamma} \to B^{t\Gamma} \to \Sigma B_{h\Gamma}$ is a fiber sequence, the pasting lemma for pullbacks identifies $\mathrm{fib}(G \to \Sigma B_{h\Gamma}) \simeq B^{h\Gamma} \times_{B^{t\Gamma}} G$, as claimed; this agrees with the inverse construction in Lemma 2.5. $\square$

Remark 2.7. The proof does not assert that $\mathcal{C} \otimes (-)$ preserves a right-lax limit presenting $\mathrm{Sp}^{g\Gamma}$. Such preservation is not formal for an arbitrary presentable $\mathcal{C}$. Instead, the argument transports the isotropy recollement, all of whose ingredients preserve colimits, and lets the Tate construction appear through the internal mapping-space computation of Lemma 2.4.

Corollary 2.8 (The C_2-Mackey presentation)**.** *There is a natural equivalence*

$$\mathrm{Mack}_\Gamma(\mathcal{C}) \simeq \mathrm{Fact}_N(\mathcal{C}).$$

Under it, B is the value on the free orbit with its Γ-action, E is the value on the fixed orbit, and the two maps of (3) are the transfer and restriction in adjoint form.

Proof. Compose (2), Theorem 2.6, and the inverse of Lemma 2.5. The identification of the values follows from the shadow and fixed-point formulas of Theorem 2.6, since in the Mackey model the free- and fixed-orbit values are uX and X^Γ. $\square$

2.4. **The symmetric monoidal structure.** Assume now that $\mathcal{C}$ is presentably symmetric monoidal and that tensor product preserves colimits separately. The Tate construction is lax symmetric monoidal [13, Sec. I.3]; write

$$\mu_{B,B'} : B^{t\Gamma} \otimes (B')^{t\Gamma} \longrightarrow (B \otimes B')^{t\Gamma}$$

for its multiplication and $\eta : \mathbf{1} \to (\mathbf{1})^{t\Gamma}$ for its unit.

Define a tensor product on fracture triples by

$$\begin{aligned} &(B, G, \gamma) \otimes (B', G', \gamma') \\ &= \left(B \otimes B',\ G \otimes G',\ G \otimes G' \xrightarrow{\gamma\otimes\gamma'} B^{t\Gamma} \otimes (B')^{t\Gamma} \xrightarrow{\mu_{B,B'}} (B \otimes B')^{t\Gamma} \right), \end{aligned} \tag{5}$$

where the Borel tensor product has diagonal Γ-action. The unit is

$$(\mathbf{1}, \mathbf{1}, \eta : \mathbf{1} \to \mathbf{1}^{t\Gamma}).$$

Proposition 2.9 (Multiplicative fracture)**.** *The tensor product (5) makes $\mathrm{Frac}_\Gamma(\mathcal{C})$ presentably symmetric monoidal, and the equivalence $\Theta_{\mathcal{C}}$ of Theorem 2.6 is symmetric monoidal.*

Proof. The pullback defining $\mathrm{Frac}_\Gamma(\mathcal{C})$ inherits a symmetric monoidal structure from the pointwise structure on $\mathrm{Fun}(\Delta^1, \mathcal{C})$, the strong monoidal ev_1, and the lax symmetric monoidal Tate construction, exactly as for the lax equalizers of [13, §II.1]; on objects it is the formula (5), with the displayed unit. On genuine objects, u and Φ^Γ are strong symmetric monoidal: both are tensored from their spectrum-level counterparts, Φ^Γ because $\widetilde{E}\Gamma$ is an idempotent E_∞-algebra, so the geometric part is a smashing symmetric monoidal localization [9, §4.8.2]. The isotropy-separation transformation $\Phi^\Gamma X \to (uX)^{t\Gamma}$

is lax symmetric monoidal: it is the mate of the connecting map of the idempotent triangle, and its multiplicativity is the E_∞-idempotence of $\widetilde{E}\Gamma$. Hence $\Theta_{\mathcal{C}}$ is lax symmetric monoidal, and its tensor and unit comparisons have equivalence components on both the Borel and geometric shadows. The forgetful functor

$$\mathrm{Frac}_\Gamma(\mathcal{C}) \longrightarrow \mathrm{Fun}(B\Gamma, \mathcal{C}) \times \mathcal{C}, \qquad (B, G, \gamma) \longmapsto (B, G),$$

is conservative, because a morphism of triples whose two shadow components are equivalences is an equivalence in the defining pullback. Therefore every comparison map is an equivalence and $\Theta_{\mathcal{C}}$ is strong symmetric monoidal. Compare the multiplicative fracture square of [12] and, for rigidly compactly generated coefficients, [1, §§5–6]. □

Corollary 2.10 (Detection of invertibility)**.** *A fracture triple is invertible only if its Borel and geometric shadows are invertible. Conversely, a morphism of fracture triples is an equivalence if and only if it is an equivalence on both shadows.*

Proof. The first assertion follows from the strong symmetric monoidality of the shadow functors. The second follows from the conservative forgetful functor used in the proof of Proposition 2.9. □

3. Tate orientations and the Picard obstruction

Let $\mathcal{C}$ be presentably symmetric monoidal and stable, with tensor product preserving colimits separately. Put

$$R_{\mathcal{C}} = (\mathbf{1}_{\mathcal{C}})^{t\Gamma} \in \mathrm{CAlg}(\mathcal{C}).$$

3.1. **Correction lines.** Let $L \in \mathrm{Pic}(\mathcal{C})$. A *framed correction object of type L* is a fracture triple with chosen identifications of its Borel shadow with $\mathbf{1}$ and its geometric shadow with L^{-1}. Thus it is represented uniquely by a map

$$\gamma : L^{-1} \longrightarrow R_{\mathcal{C}}. \tag{6}$$

It is a *framed correction line* if the associated genuine object is invertible. Let $\mathrm{Line}^{\mathrm{fr}}_L(\mathcal{C})$ denote the resulting space.

Tensoring (6) by L, followed by extension of scalars from the unit, gives an $R_{\mathcal{C}}$-linear map

$$f_\gamma : R_{\mathcal{C}} \longrightarrow L \otimes R_{\mathcal{C}}. \tag{7}$$

Theorem 3.1 (Tate-orientation classification)**.** *The assignment $\gamma \mapsto f_\gamma$ induces a natural equivalence of spaces*

$$\mathrm{Line}^{\mathrm{fr}}_L(\mathcal{C}) \simeq \mathrm{Eq}_{\mathrm{Mod}_{R_{\mathcal{C}}}(\mathcal{C})}\big(R_{\mathcal{C}}, L \otimes R_{\mathcal{C}}\big). \tag{8}$$

Proof. A map $\gamma : L^{-1} \to R_{\mathcal{C}}$ is adjoint to a map $\mathbf{1} \to L \otimes R_{\mathcal{C}}$, hence to the $R_{\mathcal{C}}$-linear map f_γ in (7).

Suppose first that f_γ is an equivalence. Its inverse is adjoint to a map $\delta : L \to R_{\mathcal{C}}$. The triples $(\mathbf{1}, L^{-1}, \gamma)$ and $(\mathbf{1}, L, \delta)$ multiply, using (5), to the unit triple. Indeed, the two unit identities are precisely the two inverse identities for f_γ. Hence the first triple is invertible.

Conversely, if $(\mathbf{1}, L^{-1}, \gamma)$ is invertible, its inverse must have Borel shadow $\mathbf{1}$ and geometric shadow L, so it has the form $(\mathbf{1}, L, \delta)$. The two product identities translate under the same adjunction into mutually inverse $R_{\mathcal{C}}$-linear maps

$$R_{\mathcal{C}} \rightleftarrows L \otimes R_{\mathcal{C}}.$$

Thus f_γ is an equivalence. The constructions are inverse on full mapping spaces: under the displayed adjunctions, both sides of (8) are identified with unions of connected components of the single space $\mathrm{Map}_{\mathcal{C}}(L^{-1}, R_{\mathcal{C}})$, and the two invertibility conditions select the same components. □

Definition 3.2. A *Tate orientation* of L is an equivalence of $R_{\mathcal{C}}$-modules

$$R_{\mathcal{C}} \xrightarrow{\simeq} L \otimes R_{\mathcal{C}}.$$

Corollary 3.3 (Existence and ambiguity)**.** *A correction line of type L exists if and only if the image of L under*

$$\mathrm{Pic}(\mathcal{C}) \longrightarrow \mathrm{Pic}(\mathrm{Mod}_{R_{\mathcal{C}}}(\mathcal{C})), \qquad L \longmapsto L \otimes R_{\mathcal{C}},$$

is trivial. When nonempty, the orientation space is a torsor under

$$\mathrm{GL}_1(R_{\mathcal{C}}) = \mathrm{Aut}_{\mathrm{Mod}_{R_{\mathcal{C}}}(\mathcal{C})}(R_{\mathcal{C}}).$$

Proof. The first assertion is Theorem 3.1. After choosing one equivalence, every other is obtained uniquely by precomposition with an automorphism of the free rank-one module. □

3.2. **Base change.**

Proposition 3.4 (Base change of Tate orientations)**.** *Let $f^* : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal colimit-preserving functor between presentably symmetric monoidal stable ∞-categories. There is a canonical map of commutative algebras*

$$f^* R_{\mathcal{C}} \longrightarrow R_{\mathcal{D}}.$$

For every $L \in \mathrm{Pic}(\mathcal{C})$, extension of scalars induces a natural map

$$\mathrm{Eq}_{\mathrm{Mod}_{R_{\mathcal{C}}}(\mathcal{C})}(R_{\mathcal{C}}, L \otimes R_{\mathcal{C}}) \longrightarrow \mathrm{Eq}_{\mathrm{Mod}_{R_{\mathcal{D}}}(\mathcal{D})}(R_{\mathcal{D}}, f^* L \otimes R_{\mathcal{D}}).$$

In particular, orientability is preserved by base change, and nonorientability after base change obstructs orientability before base change.

Proof. The functor f^* commutes with homotopy orbits and has a canonical comparison $f^*(B^{h\Gamma}) \to (f^* B)^{h\Gamma}$ for homotopy fixed points. These maps are compatible with norms, hence induce $f^*(B^{t\Gamma}) \to (f^* B)^{t\Gamma}$. For the trivial action on the tensor unit, this is the asserted algebra map; multiplicativity follows from the lax symmetric monoidality of Tate.

Given $\phi : R_{\mathcal{C}} \simeq L \otimes R_{\mathcal{C}}$, apply f^* and extend scalars along $f^* R_{\mathcal{C}} \to R_{\mathcal{D}}$:

$$R_{\mathcal{D}} \simeq f^* R_{\mathcal{C}} \otimes_{f^* R_{\mathcal{C}}} R_{\mathcal{D}} \xrightarrow{f^*\phi \otimes \mathrm{id}} f^* L \otimes R_{\mathcal{D}}.$$

This is an equivalence and is natural in ϕ. □

3.3. Vanishing away from two.

Corollary 3.5 (Away from 2)**.** *If multiplication by* 2 *is invertible in* $\mathcal{C}$*, then* $(-)^{t\Gamma} = 0$*. Consequently* $R_{\mathcal{C}} = 0$*, every line has a unique framed Tate orientation, and*

$$\mathrm{Frac}_\Gamma(\mathcal{C}) \simeq \mathrm{Fun}(B\Gamma, \mathcal{C}) \times \mathcal{C}$$

as symmetric monoidal categories.

Proof. Averaging by $\frac{1}{2}(1+\tau)$ identifies homotopy orbits and homotopy fixed points, so the norm is an equivalence. Its cofiber vanishes. The fracture gluing map is then unique, and the zero module has a contractible equivalence space with itself. $\square$

3.4. A monodromy criterion. Let H be a discrete group, let $\mathcal{D}$ be presentably symmetric monoidal and stable, and put

$$\mathcal{C} = \mathrm{Fun}(BH, \mathcal{D})$$

with pointwise tensor product. Set $A = (\mathbf{1}_{\mathcal{D}})^{t\Gamma}$. Then $R_{\mathcal{C}}$ is the constant H-local system with fiber A.

Let L_χ be a tensor-invertible local system with fiber $\mathbf{1}_{\mathcal{D}}$, classified by

$$\chi : BH \longrightarrow B\,\mathrm{GL}_1(\mathbf{1}_{\mathcal{D}}).$$

Proposition 3.6 (Monodromy criterion)**.** *The line* L_χ *is Tate-orientable if and only if the composite*

$$BH \xrightarrow{\chi} B\,\mathrm{GL}_1(\mathbf{1}_{\mathcal{D}}) \longrightarrow B\,\mathrm{GL}_1(A)$$

is nullhomotopic. In particular, a nontrivial homomorphism $H \to \pi_0\,\mathrm{GL}_1(A)$ *obstructs a Tate orientation.*

Proof. The A-module local system $L_\chi \otimes R_{\mathcal{C}}$ is classified by the displayed composite, whereas $R_{\mathcal{C}}$ is classified by the constant map. An equivalence between them is exactly a nullhomotopy of the classifying map. Apply Theorem 3.1. $\square$

Corollary 3.7 (The 2-adic sign obstruction)**.** *In* $\mathrm{Fun}(BC_2, \mathrm{Sp})$*, the sphere with sign monodromy is not Tate-orientable.*

Proof. Lin's theorem identifies

$$A = \mathbb{S}^{tC_2} \simeq \mathbb{S}_2^\wedge, \qquad \pi_0\,\mathrm{GL}_1(A) = \mathbb{Z}_2^\times$$

[8]. Sign monodromy maps the generator to the nontrivial unit -1, so Proposition 3.6 applies. $\square$

4. Genuine coefficients in Berkovich motives

Let X be a small arc-stack. We use the notation from the introduction:

$$\mathcal{C}_X^{\mathrm{eff}} = \mathcal{D}_{\mathrm{mot}}^{\mathrm{eff}}(X; \mathrm{Sp}), \qquad \mathcal{C}_X = \mathcal{D}_{\mathrm{mot}}(X; \mathrm{Sp}), \qquad \mathbb{T}_X \in \mathcal{C}_X^{\mathrm{eff}}.$$

4.1. Coefficient extension and cancellation.

Proposition 4.1 (Full faithfulness after coefficient extension)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful colimit-preserving functor of presentable stable ∞-categories. If $\mathcal{E}$ is compactly generated and stable, then*

$$F \otimes \mathrm{id}_{\mathcal{E}} : \mathcal{A} \otimes \mathcal{E} \longrightarrow \mathcal{B} \otimes \mathcal{E}$$

is fully faithful.

Proof. Write $\mathcal{E} \simeq \mathrm{Ind}(\mathcal{E}^{\omega})$. There are natural equivalences

$$\mathcal{A} \otimes \mathcal{E} \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{E}^{\omega})^{\mathrm{op}}, \mathcal{A}), \qquad \mathcal{B} \otimes \mathcal{E} \simeq \mathrm{Fun}^{\mathrm{ex}}((\mathcal{E}^{\omega})^{\mathrm{op}}, \mathcal{B})$$

[9, Sec. 4.8]. The induced functor is postcomposition with F. Mapping spaces of natural transformations are limits of pointwise mapping spaces, and F is an equivalence on every pointwise mapping space. Hence postcomposition is fully faithful. □

Corollary 4.2 (Coefficientwise cancellation)**.** *For every compactly generated stable coefficient category $\mathcal{E}$, the functor*

$$- \otimes \mathbb{T}_X : \mathcal{C}_X^{\mathrm{eff}} \otimes \mathcal{E} \longrightarrow \mathcal{C}_X^{\mathrm{eff}} \otimes \mathcal{E}$$

is fully faithful. In particular this holds for $\mathcal{E} = \mathrm{Sp}^{g\Gamma}$, and the coefficientwise effective-to-stable functor

$$j_X^g : \mathcal{C}_X^{\mathrm{eff},g} \longrightarrow \mathcal{C}_X^g$$

is fully faithful and strong symmetric monoidal.

Proof. Apply Proposition 4.1 first to Scholze's cancellation functor (1), and then to the fully faithful effective-to-stable functor supplied by cancellation. The category $\mathrm{Sp}^{g\Gamma}$ is compactly generated. □

4.2. The canonical coefficientwise object.

Let

$$u : \mathrm{Sp}^{g\Gamma} \longrightarrow \mathrm{Fun}(B\Gamma, \mathrm{Sp}), \qquad \Phi^{\Gamma} : \mathrm{Sp}^{g\Gamma} \longrightarrow \mathrm{Sp}$$

be the Borel and geometric-fixed-point functors. For the virtual representation sphere,

$$uS^{\sigma-1} \simeq S^0_{\mathrm{sgn}}, \qquad \Phi^{\Gamma} S^{\sigma-1} \simeq S^{-1}. \tag{9}$$

Here S^0_{sgn} is the sphere on which the generator acts by the degree -1 self-equivalence.

Definition 4.3. The *canonical coefficientwise genuine Tate object* is

$$\mathbb{T}_X^{\mathrm{cw}} = \mathbb{T}_X \boxtimes S^{\sigma-1} \in \mathcal{C}_X^{\mathrm{eff},g}.$$

Theorem 4.4 (Canonical shadows and cancellation)**.** *There are natural equivalences*

$$u\mathbb{T}_X^{\mathrm{cw}} \simeq \mathbb{T}_X \boxtimes S^0_{\mathrm{sgn}}, \qquad \Phi^{\Gamma}\mathbb{T}_X^{\mathrm{cw}} \simeq \mathbb{T}_X[-1].$$

The stable image of $\mathbb{T}_X^{\mathrm{cw}}$ is invertible, and

$$- \otimes \mathbb{T}_X^{\mathrm{cw}} : \mathcal{C}_X^{\mathrm{eff},g} \longrightarrow \mathcal{C}_X^{\mathrm{eff},g}$$

is fully faithful.

Proof. The shadow formulas follow from (9) and the symmetric monoidality of u and Φ^Γ. In $\mathcal{C}_X$, the object $\mathbb{T}_X$ is invertible by definition, and $S^{\sigma-1}$ is invertible in $\mathrm{Sp}^{g\Gamma}$; hence the stable image is invertible. Tensoring by $S^{\sigma-1}$ is already an autoequivalence of the coefficient category, while tensoring by $\mathbb{T}_X$ is fully faithful by Corollary 4.2. $\square$

4.3. **Inversion and additive sign.** The additive sign action in Theorem 4.4 should not be confused with an involution induced by inversion on a group object. An equality $[\iota] = -[\mathrm{id}]$ in a homotopy category controls only π_0 of a mapping space and does not, by itself, produce an equivalence of coherent C_2-actions.

Lemma 4.5 (Antipode identity)**.** *Let H be an augmented commutative Hopf algebra object in an additive symmetric monoidal homotopy category. Let I be the augmentation summand, $\overline{\Delta} : I \to I \otimes I$ the reduced comultiplication, and S the antipode. Then*

$$S = -\,\mathrm{id}_I - \mu(S \otimes \mathrm{id})\overline{\Delta}.$$

Thus $\overline{\Delta} = 0$ implies the equality $[S] = -[\mathrm{id}_I]$, but not automatically an equivalence of coherent C_2-objects.

Proof. For $x \in I$, write $\Delta(x) = x \otimes 1 + 1 \otimes x + \overline{\Delta}(x)$. The antipode identity $\mu(S \otimes \mathrm{id})\Delta = \eta\epsilon$ has zero right-hand side on I. Expanding gives the formula. $\square$

Remark 4.6 (Contrast with $\mathbb{A}^1$-homotopy theory). The hypothesis $\overline{\Delta} = 0$ is not formal. In the motivic stable category $\mathrm{SH}(k)$ of a field, applied to $\mathbb{G}_m$, inversion acts on the Tate sphere by $-\langle -1 \rangle = -1 - \eta[-1]$, so the correction term equals $\eta[-1]$, which is nonzero for instance over $k = \mathbb{R}$ [11, §3]. Both the weak and the coherent comparison are therefore statements about the particular motivic theory, not formal properties of a group object. No result of this paper assumes either: all normalizations are stated through the additive sign object $\mathbb{T}_X \boxtimes S^0_{\mathrm{sgn}}$, and the real obstruction of §6 uses only π_0-monodromy (Remark 6.4).

Proposition 4.7 (Coherent comparison at the complex point)**.** *At $X = \mathcal{M}_{\mathrm{arc}}(\mathbb{C})$, inversion on the first Tate object is coherently equivalent to the additive sign action.*

Proof. Under Scholze's complex comparison, the first Tate object is the desuspended reduced suspension spectrum of $\mathbb{C}^\times$. The deformation retraction

$$\mathbb{C}^\times \longrightarrow U(1), \qquad z \longmapsto z/|z|,$$

is equivariant for inversion. On $U(1) \cong S^1$, inversion is reflection and has degree -1. After desuspension this gives an equivalence of the resulting coherent C_2-object with S^0_{sgn}. $\square$

5. Normalized genuine Tate lines

Put

$$R_X = (\mathbf{1}_X)^{t\Gamma} \in \mathrm{CAlg}(\mathcal{C}_X).$$

A *framed normalized stable genuine Tate line* is an invertible object $\mathbb{L}_X \in \mathcal{C}_X^g$, together with chosen equivalences

$$u\mathbb{L}_X \simeq \mathbb{T}_X \boxtimes S^0_{\mathrm{sgn}}, \qquad \Phi^\Gamma \mathbb{L}_X \simeq \mathbf{1}_X[-1].$$

The framing is part of the datum.

Theorem 5.1 (Normalized-line classification)**.** *The space of framed normalized stable genuine Tate lines over X is naturally equivalent to*

$$\mathrm{Eq}_{\mathrm{Mod}_{R_X}(\mathcal{C}_X)}\big(R_X, \mathbb{T}_X \otimes R_X\big). \tag{10}$$

Hence a normalized stable line exists if and only if $\mathbb{T}_X$ is Tate-orientable. If this space is nonempty, it is a torsor under $\mathrm{GL}_1(R_X)$.

Proof. The stable image of $\mathbb{T}_X^{\mathrm{cw}}$ is invertible and has geometric shadow $\mathbb{T}_X[-1]$. Tensoring a normalized line by its inverse produces a framed correction line with Borel shadow $\mathbf{1}_X$ and geometric shadow $\mathbb{T}_X^{-1}$. This operation is reversible. Apply Theorem 3.1 with $L = \mathbb{T}_X$. □

Remark 5.2. Theorem 5.1 is a classification, not an existence assertion. In particular, stable invertibility of $\mathbb{T}_X^{\mathrm{cw}}$ does not trivialize $\mathbb{T}_X \otimes R_X$, and formally adjoining $\mathbb{T}_X^{-1}$ does not solve the effective normalization problem.

6. The real-point obstruction

Let $X_\mathbb{R} = \mathcal{M}_{\mathrm{arc}}(\mathbb{R})$ and $X_\mathbb{C} = \mathcal{M}_{\mathrm{arc}}(\mathbb{C})$. Complex conjugation generates $G_\mathbb{R} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We continue to reserve Γ for the independent coefficient group used in genuine spectra and in the Tate construction.

6.1. Real Betti–Galois realization.

Proposition 6.1 (Real Betti–Galois realization)**.** *There is a symmetric monoidal colimit-preserving functor*

$$\mathrm{Re}_{B,\mathbb{R}} : \mathcal{C}_{X_\mathbb{R}} \longrightarrow \mathrm{Fun}(BG_\mathbb{R}, \mathrm{Sp})$$

which sends $\mathbb{T}_\mathbb{R}$ to the sphere with sign $G_\mathbb{R}$-monodromy.

Proof. The map $c : X_\mathbb{C} \to X_\mathbb{R}$ gives a symmetric monoidal pullback $c^* : \mathcal{C}_{X_\mathbb{R}} \to \mathcal{C}_{X_\mathbb{C}}$. Complex conjugation generates a $G_\mathbb{R}$-action on $X_\mathbb{C}$ over $X_\mathbb{R}$, so every pullback c^*M carries a canonical coherent $G_\mathbb{R}$-equivariant structure, and pullback refines to

$$\mathcal{C}_{X_\mathbb{R}} \longrightarrow \big(\mathcal{C}_{X_\mathbb{C}}\big)^{hG_\mathbb{R}}.$$

This refinement uses only the functoriality of $X \mapsto \mathcal{C}_X$ in morphisms of small arc-stacks, not any descent property; the target is a limit of a $BG_\mathbb{R}$-diagram of presentable categories with colimit-preserving transition functors, so the refined functor is symmetric monoidal and colimit-preserving because its components are. Scholze's archimedean comparison identifies the sphere-valued category $\mathcal{C}_{X_{\mathbb{C}}}$ symmetrically monoidally with Sp and identifies $\mathbb{T}_{\mathbb{C}}$ with the tensor unit [14, Prop. 1.5(i), Prop. 3.9, Def. 5.18]. The space of colimit-preserving symmetric monoidal autoequivalences of Sp is contractible, Sp being initial among presentably symmetric monoidal stable ∞-categories [9, §4.8.2], so the induced action on the coefficient category is canonically trivial. Hence the target identifies with $\mathrm{Fun}(BG_{\mathbb{R}}, \mathrm{Sp})$.

It remains to compute the descent action on $\mathbb{T}_{\mathbb{C}}$. In the complex comparison, $\mathbb{T}_{\mathbb{C}}$ is the desuspended reduced suspension spectrum of $\mathbb{C}^{\times}$. Complex conjugation preserves the equivariant deformation retraction $\mathbb{C}^{\times} \to U(1)$, and on $U(1) \cong S^1$ it is reflection, of degree -1. After desuspension the resulting local system is the sphere with sign monodromy. □

Remark 6.2. Only the realization functor of Proposition 6.1 is needed. No equivalence between all real Berkovich motives and naive $G_{\mathbb{R}}$-spectra is asserted.

6.2. **Nonorientability.**

Theorem 6.3 (Real obstruction). *The first Tate object $\mathbb{T}_{\mathbb{R}}$ is not Tate-orientable. Consequently there is no normalized stable genuine Tate line over $X_{\mathbb{R}}$, and therefore no effective one.*

Proof. Assume that $\mathbb{T}_{\mathbb{R}}$ has a Tate orientation. By Proposition 3.4, its image under $\mathrm{Re}_{B,\mathbb{R}}$ would be Tate-orientable in $\mathrm{Fun}(BG_{\mathbb{R}}, \mathrm{Sp})$. Proposition 6.1 identifies that image with the sign local system. Corollary 3.7, applied with the descent group $G_{\mathbb{R}}$ and the separate coefficient group Γ, says that this local system is not Tate-orientable. This contradiction proves the first assertion. The stable nonexistence statement follows from Theorem 5.1; an effective normalized object with invertible stable image would supply such a stable line. □

The preceding proof can be written concretely. In $\mathrm{Fun}(BG_{\mathbb{R}}, \mathrm{Sp})$, the Tate algebra is the constant local system with fiber

$$\mathbb{S}^{t\Gamma} \simeq \mathbb{S}_2^{\wedge}.$$

An equivalence from this constant free rank-one module to its sign twist would be represented on π_0 by a unit $a \in \mathbb{Z}_2^{\times}$. Equivariance requires $a = -a$, hence $2a = 0$, impossible in $\mathbb{Z}_2$.

Remark 6.4 (Coherence-independence). The argument uses only the π_0-monodromy of the realized Tate object: the generator acts by -1 on the fiber, whatever higher coherence the action carries. The distinctions between weak and coherent sign comparisons drawn in §4 therefore do not affect Theorem 6.3 or Theorem 6.5 below.

Theorem 6.5 (Bases with a real point). *Let X be a small arc-stack admitting a morphism*

$$x : \mathcal{M}_{\mathrm{arc}}(\mathbb{R}) \longrightarrow X.$$

Then $\mathbb{T}_X$ is not Tate-orientable. In particular, X admits no normalized stable or effective genuine Tate line.

Proof. Pullback along x is symmetric monoidal and colimit-preserving and carries $\mathbb{T}_X$ to $\mathbb{T}_{\mathbb{R}}$. An orientation over X would therefore restrict, by Proposition 3.4, to an orientation over the real point, contradicting Theorem 6.3. □

Corollary 6.6. *Any arithmetic arc-stack that contains the archimedean real point, including $\mathcal{M}_{\mathrm{arc}}(\mathbb{Z})$ through its usual real norm, is nonorientable.*

Proof. Apply Theorem 6.5 to the corresponding real point. □

7. Effective objects and positive regimes

The stable classification is complete, but an orientation in $\mathcal{C}_X$ need not come from an object of $\mathcal{C}_X^{\mathrm{eff}}$. We now state exactly what is required.

Proposition 7.1 (Stable invertibility implies effective cancellation)**.** *Let $j : \mathcal{A} \to \mathcal{B}$ be fully faithful and strong symmetric monoidal. If $M \in \mathcal{A}$ has invertible image jM, then*

$$- \otimes M : \mathcal{A} \longrightarrow \mathcal{A}$$

is fully faithful.

Proof. For $P, Q \in \mathcal{A}$,

$$\begin{aligned} \mathrm{Map}_{\mathcal{A}}(P, Q) &\simeq \mathrm{Map}_{\mathcal{B}}(jP, jQ) \\ &\simeq \mathrm{Map}_{\mathcal{B}}(jP \otimes jM, jQ \otimes jM) \\ &\simeq \mathrm{Map}_{\mathcal{A}}(P \otimes M, Q \otimes M). \end{aligned}$$

□

Proposition 7.2 (Exact effective-lifting criterion)**.** *Suppose $\mathbb{T}_X$ is Tate-orientable, and let $\mathbb{L}_X \in \mathcal{C}_X^g$ be the associated framed normalized stable line. An effective normalized object lifting $\mathbb{L}_X$ exists if and only if all of the following hold:*

(i) *$\mathbb{L}_X$ lies in the essential image of $j_X^g : \mathcal{C}_X^{\mathrm{eff},g} \to \mathcal{C}_X^g$;*
(ii) *the prescribed Borel and geometric target objects are effective;*
(iii) *the chosen shadow identifications and their coherence homotopies lie in the corresponding effective mapping spaces.*

When these conditions hold, full faithfulness of j_X^g and of its shadow-wise analogues makes the effective lift, together with all framed data, unique up to a contractible space of choices. Tensoring with any such effective lift is fully faithful.

Proof. Necessity is immediate. Conversely, choose an effective preimage of $\mathbb{L}_X$ using (i). Corollary 4.2 makes j_X^g fully faithful, and the same coefficient-extension argument applies to the Borel and geometric shadow categories. Thus the maps and homotopies in (iii) descend precisely when they lie in the stated effective mapping spaces. The stable image of the resulting effective object is invertible, so Proposition 7.1 gives full faithfulness of tensoring. □

Remark 7.3. Proposition 7.2 is not an assumed lemma in the proof of the stable theorems. It is a complete statement of the additional input required to pass from a stable orientation to an effective normalized object. Separate effective lifts of the two shadows do not suffice: the genuine gluing map must lift as well.

7.1. **The complex point.**

Corollary 7.4 (Complex point)**.** *For $X = \mathcal{M}_{\mathrm{arc}}(\mathbb{C})$, the canonical object $\mathbb{T}_X^{\mathrm{cw}}$ is already normalized, effectively and stably.*

Proof. Scholze's complex comparison identifies $\mathbb{T}_X$ with $\mathbf{1}_X$ already in the effective category. The geometric shadow in Theorem 4.4 is therefore $\mathbf{1}_X[-1]$. ☐

7.2. **After inverting two.**

Corollary 7.5 (Effective normalization away from 2)**.** *After inverting* 2*, every small arc-stack admits a canonical effective normalized object whose stable image is invertible. Tensoring with this object is fully faithful.*

Proof. Apply Corollary 3.5 to the effective category $\mathcal{C}_X^{\mathrm{eff}}[1/2]$. The coefficientwise fracture category splits, so the pair

$$\left(\mathbb{T}_X \boxtimes S^0_{\mathrm{sgn}},\ \mathbf{1}_X[-1]\right)$$

defines an effective genuine object with the desired shadows. In the split stable fracture category its inverse is the pair of the inverse Borel and geometric components. Proposition 7.1 gives effective cancellation. ☐

7.3. **Residue characteristic two.**

Corollary 7.6 (Geometric points of residue characteristic 2)**.** *Let $X = \mathcal{M}_{\mathrm{arc}}(C)$, where C is an algebraically closed nondiscrete nonarchimedean Banach field of residue characteristic* 2*. Then $R_X = 0$, $\mathbb{T}_X$ has a unique framed Tate orientation, and X admits an effective normalized genuine Tate object whose stable image is invertible.*

Proof. Scholze proves that the residue characteristic is invertible in the effective motivic category at such a geometric point [14, Prop. 1.5(iii)]. Thus 2 is invertible and Corollary 7.5 applies. Equivalently, $R_X = 0$ and the equivalence space between the two zero R_X-modules is contractible. ☐

Remark 7.7. The proof does not use a reduced two-point μ_2-fixed locus, which degenerates in characteristic 2. The obstruction disappears instead because 2 is already invertible in the local motivic category.

7.4. **The remaining locus problem.**

Problem 7.8 (Tate-oriented and effective loci)**.** For geometrically meaningful classes of small arc-stacks X, determine:

(i) the class of $\mathbb{T}_X \otimes R_X$ in $\mathrm{Pic}(\mathrm{Mod}_{R_X}(\mathcal{C}_X))$;

(ii) the space of Tate orientations when that class is trivial;
(iii) which oriented stable lines satisfy the effective-lifting criterion of Proposition 7.2;
(iv) the behavior of these choices under descent and base change;
(v) the sphere-level obstruction at nonarchimedean geometric points of odd or zero residue characteristic.

Scholze's local calculations identify several linearized nonarchimedean motivic categories, including the relevant 2-power torsion categories in suitable geometric cases. Translating those calculations into the required sphere-level equivalence of R_X-modules is a separate problem and is not asserted here.

8. Concluding perspective

The canonical coefficientwise object $\mathbb{T}_X^{\mathrm{cw}}$ and its integral effective cancellation are unconditional formal consequences of Scholze's cancellation theorem and genuine representation spheres. The normalized object is subtler: its existence is governed by the Tate algebra $R_X = (\mathbf{1}_X)^{tC_2}$ and the image of $\mathbb{T}_X$ in the Picard group of R_X-modules. This obstruction is invisible after inverting 2, but integral information survives through the 2-complete Tate algebra.

The real point shows that the universal normalized-line conjecture is false for a structural reason rather than because of a missing technical lemma. At the same time, the classification theorem turns the failed universal statement into a concrete geometric program: compute the Tate-oriented locus and then solve the independent effective-lifting problem on that locus. The fracture description provides the correct language for both tasks, because it retains the coherent gluing datum that is lost when one records only Borel and geometric shadows.

References

[1] D. Ayala, A. Mazel-Gee, and N. Rozenblyum, *Derived Mackey functors and C_{p^n}-equivariant cohomology*, arXiv:2105.02456 (2021).

[2] D. Ayala, A. Mazel-Gee, and N. Rozenblyum, *Stratified noncommutative geometry*, Mem. Amer. Math. Soc. **297** (2024), no. 1485.

[3] C. Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Adv. Math. **304** (2017), 646–727.

[4] B. Calmès, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle, *Hermitian K-theory for stable ∞-categories I: Foundations*, Selecta Math. (N.S.) **29** (2023), Paper No. 10.

[5] B. Calmès, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle, *Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity*, Acta Math. **235** (2025), no. 2, 149–400.

[6] A. Krause, *The Picard group in equivariant homotopy theory via stable module categories*, J. Topol. **18** (2025), no. 2, Paper No. e70020.

[7] L. G. Lewis, Jr., J. P. May, and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer, Berlin, 1986.

[8] W.-H. Lin, *On conjectures of Mahowald, Segal and Sullivan*, Math. Proc. Cambridge Philos. Soc. **87** (1980), no. 3, 449–458.

[9] J. Lurie, *Higher Algebra*, available from the author's webpage, 2017.
[10] A. Mathew, N. Naumann, and J. Noel, *Nilpotence and descent in equivariant stable homotopy theory*, Adv. Math. **305** (2017), 994–1084.
[11] F. Morel, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012.
[12] N. Naumann, L. Pol, and M. Ramzi, *A symmetric monoidal fracture square*, arXiv:2411.05467 (2024).
[13] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Math. **221** (2018), no. 2, 203–409.
[14] P. Scholze, *Berkovich motives*, J. Amer. Math. Soc. **39** (2026), no. 3, 697–764; arXiv:2412.03382.

Columbia University, New York, NY, USA
Email address: `rc3148@columbia.edu`